# Synchronization on complex networks

## A model for neural networks

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18 April 2018

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If you are looking for your next popular science book to read try: 'Sync: The emerging science of spontaneous order' - Steven Strogatz

# § YouTube Video

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### Example: Coin flipping

- win 1 € if heads
- lose 1 € if tails

Exercise:

$$\omega = \{H, T, T, T, T, H, T, H \ldots\}$$

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Oscillators evolve according to a system of coupled stochastic differential equations

$$\mathrm{d}\theta_i(t) = \frac{K}{N}\sum_{j=1}^{N} \sin\left[\theta_j(t) - \theta_i(t)\right]\mathrm{d}t + D\,\mathrm{d}W_i(t). \qquad (1)$$

Here, $K \in (0,\infty)$ is the interaction strength, $D \in (0,\infty)$ is the noise strength, and $(W_i(t))_{t\geq 0}$ are noise processes.

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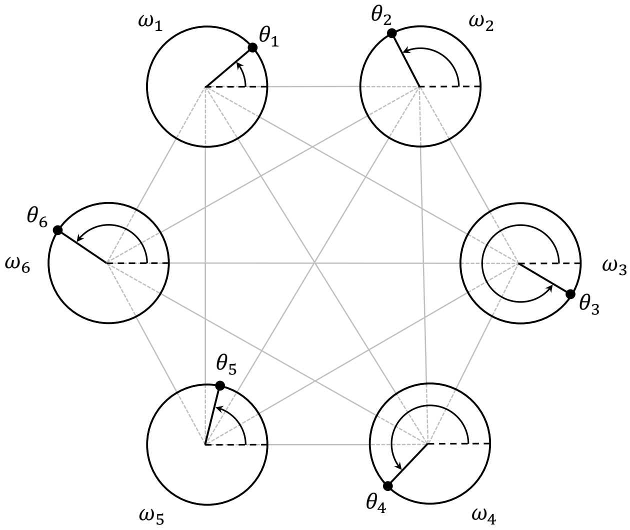
**Question:**

Can you spot the network here?

# Cartoon of the Kuramoto model for $N = 6$



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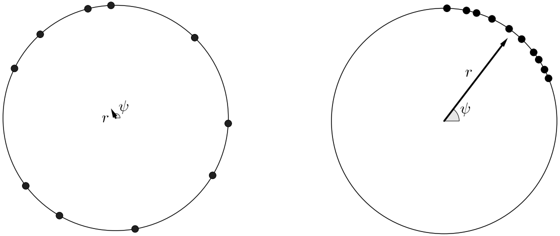


Phase distributions with $r = 0.095$ and $r = 0.929$.

# Taking limits

Rewriting using the order parameter (exercise) gives

$$\mathrm{d}\theta_i(t) = K\, r_N(t) \sin\left[\psi_N(t) - \theta_i(t)\right] \mathrm{d}t + D\, \mathrm{d}W_i(t), \qquad (3)$$

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As $N$ gets ever larger, you can describe the evolution of the oscillators as the evolution of a density.

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**The large time limit (steady-state)**

Question: Does the density of the system stop evolving at some point?

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There exists a critical threshold $K_c$ such that:

(I) For $K < K_c$ the system relaxes to an unsynchronized state $(r = 0)$.

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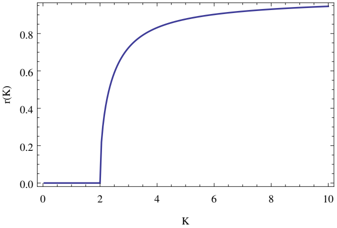
**Theorem**

$$K_c = 2 \tag{4}$$



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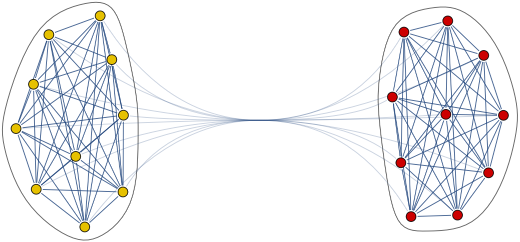

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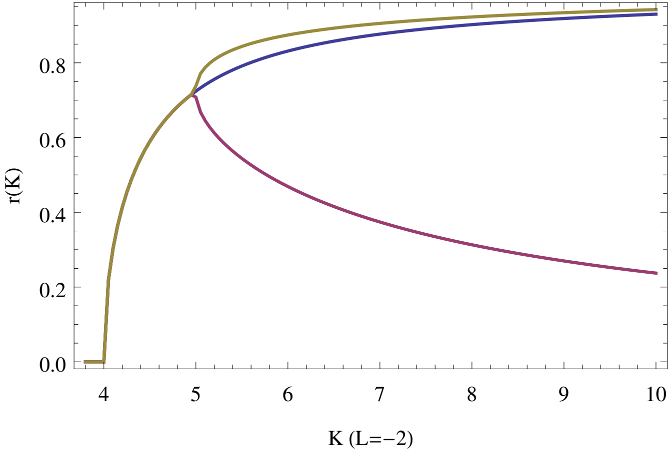

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- Using mathematics we can tell neuroscientists something of value
- Inter-disciplinary research is becoming more and more important