NETWORKS GOES TO SCHOOL #3

for secondary schools

NET WORKS

NETWORKS

NETWORKS is a 10-year research programme hosted by the University of Amsterdam (UvA), Eindhoven University of Technology (TU/e), Leiden University (UL), and the Center for Mathematics and Computer Science (CWI) in Amsterdam.

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INTRODUCTION

In March 2020 the third “NETWORKS goes to school” event was organised. This Masterclass, organised by the NWO Gravitation programme NETWORKS, is the third edition. The aim of these events is to provide secondary education students and teachers a first mathematical introduction on network science. This book collects the material realised for this third “NETWORKS goes to school” event.

In Chapter 1 all the necessary background material that is required for Chapter 2 is presented. In Section 2.1, we introduce queueing theory by showing how to model and analyse a queueing system with standard techniques. Section 2.2 focuses on road traffic networks, and discusses how navigation systems select routes. Chapter 3 contains exercises on these two topics and in Chapter 4 we provide the corresponding solutions. Chapters 2, 3 and 4 were written with the help of Rens Kamphuis (University of Amsterdam) and Youri Raaijmakers (Eindhoven University of Technology).

For more information and the books of the first two masterclasses “NETWORKS goes to school”, please visit www.networkpages.nl.

On behalf of the NETWORKS programme,

the organising committee of “NETWORKS goes to school”
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CHAPTER 1

Mathematical background

In this chapter the necessary background knowledge is provided. In Section 1.2 some basic concepts from probability theory together with some examples are presented. In Section 1.3 some basic concepts from graph theory and the theory of algorithms are presented. Dijkstra's algorithm to find the shortest route in a network is discussed.

1.1. Basic notation

We start by introducing some notation we will use in the sequel:

(1) $\mathbb{N}$ for the set of natural numbers, that is $\mathbb{N} = \{1, 2, 3, \cdots\}$;

(2) $\mathbb{Z}$ for the set of integer numbers, that is $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.

(3) $\mathbb{R}$ for the set of real numbers, that is all integer numbers and all the decimal numbers between them.

1.2. Probability theory

Probability theory is the area of mathematics that studies random phenomena. For example if the experiment is tossing a coin, then there are two possible outcomes, either *heads* or *tails*. Each outcome occurs with probability 0.5. In order to study such a random experiment we use random variables.

Random variable

A **random variable** X is a variable whose possible values are outcomes of a random experiment. We will also use the term *stochastic* as a synonym for random.

We define a random variable by giving the *state space*, i.e. the set of all possible values the variable can take, and the *probability function*, which yields the corresponding probability that a given outcome will occur. For the coin toss for example we can define a random variable by assigning to the outcome *heads* the value 1 and to the outcome *tails* the value 0. In this case we have

$$X(\textit{heads}) = 1 \quad \text{and} \quad X(\textit{tails}) = 0.$$

The probability function for this random variable is given by

$$\mathbb{P}(X = 1) = \mathbb{P}(\textit{heads}) = 0.5,$$

and

$$\mathbb{P}(X = 0) = \mathbb{P}(\textit{tails}) = 0.5,$$

where for a possible set of outcomes A, $\mathbb{P}(A)$ denotes the probability that A occurs. A random variable can be *discrete* or *continuous*.

Discrete random variables

A random variable X is called discrete when it can take countable many values, for simplicity we can just say that its values are the integer numbers, that is $X \in \mathbb{Z}$.

Continuous random variables

A random variable X is called continuous when it can take continuously many values, for simplicity we can just say that its values are the real numbers that is $X \in \mathbb{R}$.

For a discrete random variable, we can write down the probability that it equals a specific value. For a continuous random variable, this is not possible, as there is a continuum of possible values. We can however specify the probability that a continuous random variable falls in a range of values by using the **density function**. The probability that a continuous random variable X assumes values in the interval $[a, b]$ is given by the integral of the density function, denoted by f_X, over that interval:

$$\int_a^b f_X(x)\mathrm{d}x = \mathbb{P}(X \in [a, b]).$$

The result of this integration gives the area delimited by the graph of the density function f_X, the x-axis and the vertical lines given by $y = a$ and $y = b$.

Expectation of a random variable

For a random variable X, discrete or continuous, we define the expectation, or expected value, as the average of a large number of independent realisations of the random variable. We denote the expectation of X by $\mathbb{E}[X]$.

For a discrete random variable its expectation is defined by

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k\mathbb{P}(X = k). \tag{1.2.1}$$

For a continuous random variable its expectation is defined by

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x, \tag{1.2.2}$$

where f_X denotes the density function of the random variable, this means that

$$f_X(x)\mathrm{d}x = \mathbb{P}(X \in \mathrm{d}x). \tag{1.2.3}$$

As we will see in the sequel, if the random variable takes only positive values then the integral in the expectation starts from 0 instead of $-\infty$.

1.2.1. Bernoulli random variable

Bernoulli random variable

A **Bernoulli random variable** describes the outcome of any single random experiment that asks a yes-no question, like tossing a coin.

It takes the value 1 with probability p and the value 0 with probability $1-p$. Consider for example a coin where one side is heavier, then this is a biased coin where one side is favoured. We will use $B(p)$ to denote a Bernoulli random variable with probability p. A Bernoulli random variable has expectation given by

$$\mathbb{E}[B(p)] = 1 \cdot \mathbb{P}(B(p) = 1) + 0 \cdot \mathbb{P}(B(p) = 0) = p. \tag{1.2.4}$$

1.2.2. Binomial random variable

Binomial random variable

A **binomial random variable** describes the number of successes in a sequence of independent experiments, each asking a yes–no question.

We make the following assumptions:

- the number n of observations is fixed;
- each observation is independent of the other observations;
- each observation represents one of two outcomes: success or failure (yes-no);
- the probability p of success is exactly the same for each trial.

Under these assumptions, we can describe each binomial random variable by using the parameters n and p. We will denote a binomial random variable by $B(n,p)$. A binomial random variable has state space $\{0, 1, \dots, n\}$, and the probability that $B(n,p)$ is equal to k is given by

$$\mathbb{P}(B(n,p) = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is the **binomial coefficient**. The symbol $\binom{n}{k}$ is read as 'n choose k', as this is the number of ways to choose k different elements from a total of n elements, where the order of elements does not matter. The factorial of n is denoted by $n!$ and it is equal to the product $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$. The binomial random variable has expectation equal to

$$\mathbb{E}[B(n,p)] = \sum_{k=0}^{\infty} k \mathbb{P}(B(n,p) = k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np. \tag{1.2.5}$$

The exact derivation of this result is far away from the scope of this booklet.

EXAMPLE 1.2.1. Suppose that we have a total of 5 colours, and we wish to know how many combinations there are of 3 different colours, where the order of the colours does not matter. Then $n = 5$ and $k = 3$, and

$$\binom{5}{3} = \frac{5!}{3!2!} = 10.$$

We could also reason in a different way. For the first choice we have a total of 5 possible colours, for the second choice we have 4 possible colours and for the third choice we have 3 possible colours. The total of combinations of three colours is then $5 \cdot 4 \cdot 3 = 5!/2!$. However, the order of colours did not matter so we still have to divide by the number of ways in which we can order 3 colours, which is $3 \cdot 2 \cdot 1 = 3!$.

EXAMPLE 1.2.2. Consider a coin toss, where possible outcomes are heads or tails. Suppose that we have a fair coin, i.e., the probability for heads is the same as it is for tails. If we toss the coin 10 times, then the number of coin tosses that came heads from those ten tosses has a binomial distribution with parameters $n = 10$ and $p = \frac{1}{2}$. The probability of getting exactly four heads is equal to

$$\mathbb{P}(X = 4) = \binom{10}{4} \frac{1}{2}^4 \left(1 - \frac{1}{2}\right)^{10-4} = \frac{105}{512} \approx 0.205.$$

1.2.3. Geometric random variable

Geometric random variable

A **geometric random variable** describes the number of failures in a sequence of random experiments, each asking a yes-no question, until the first success.

We make the following assumptions:

- each observation is independent of the other observations;
- each observation represents one of two outcomes: success or failure;
- the probability p of success is exactly the same for each trial.

Under these assumptions, we can describe each geometric distribution by using the parameter p, we will denote a geometric random variable by $G(p)$. The geometric random variable has state space $\{0, 1, 2, \ldots\}$, and the probability that $G(p)$ is equal to k is given by

$$\mathbb{P}(G(p) = k) = (1 - p)^k p.$$

When the random variable $G(p)$ is equal to k then we know that k failures have occurred before the first success. The probability of a failure is equal to $1 - p$ and by the assumptions above the experiments we perform are independent of each other. The geometric random variable has expectation equal to

$$\mathbb{E}[G(p)] = \sum_{k=0}^{\infty} k\mathbb{P}(G(p) = k) = \sum_{k=0}^{\infty} k(1 - p)^k p = \frac{1 - p}{p}. \qquad (1.2.6)$$

Again the exact derivation of the formula is far away from the scope of this booklet. For some more details on this formula we refer to the solution of Exercise 4.2.1.

EXAMPLE 1.2.3. Consider a coin toss, where possible outcomes are heads or tails. Suppose that we have an unfair coin, i.e., the probability for heads is $\frac{1}{3}$ and the probability for tails is $\frac{2}{3}$. Then the probability to get five times tails before the first heads is equal to

$$\mathbb{P}\left(G\left(\frac{1}{3}\right)=5\right)=\left(\frac{2}{3}\right)^5\frac{1}{3}\approx 0.044.$$

1.2.4. Exponential random variable

Exponential random variable

The **exponential random variable** is a continuous random variable and describes the time elapsed between events that occur continuously and independently at a constant intensity.

An exponential random variable is characterised by a parameter λ, called the intensity. The larger this parameter is the higher the frequency of the arriving events. A random variable having the exponential distribution with parameter λ, denoted by $E(\lambda)$, has the following probability distribution function

$$\mathbb{P}(E(\lambda)\leq x)=1-e^{-\lambda x},\qquad x\geq 0, \tag{1.2.7}$$

and a probability density function given by

$$f_\lambda(x)=\lambda e^{-\lambda x},\qquad x\geq 0. \tag{1.2.8}$$

The expectation of the exponential random variable is equal to

$$\mathbb{E}[E(\lambda)]=\int_0^\infty x f_{E(\lambda)}(x)\mathrm{d}x=\int_0^\infty x\lambda e^{-\lambda x}\mathrm{d}x=\frac{1}{\lambda}. \tag{1.2.9}$$

The exponential random variable has the memoryless property, i.e. that means that

$$\mathbb{P}(E(\lambda)>x+y|E(\lambda)>y)=\mathbb{P}(E(\lambda)>x),\qquad x,y\geq 0. \tag{1.2.10}$$

The probability on the left-hand side in the equation above is called a *conditional probability*, for more details we refer to Section 3.1.1. This memoryless property is quite remarkable, so let's look at it from a practical side. Suppose the time until the bus arrives is exponentially distributed. If that would be the case, then if the bus didn't arrive for an hour, then it would still take the same amount of time until the bus arrives. But in reality we expect that if the bus didn't arrive for an hour, then it will probably arrive soon.

1.2.5. Poisson process

Finally, we introduce the Poisson process. This represents a sequence of events where events happen once every while. The time between events is exponentially distributed. Since the exponential distribution is memoryless, the Poisson process has a very remarkable property. If no event happened for a while, it doesn't imply that some event will occur soon. As an example, consider the time until you hit a specific number on a roulette wheel. If that specific number didn't show up for a while, that doesn't make it more likely for the number to show up sooner than normal. In other words: the history of the process has no influence on the future.

1.2.6. Normal random variable

Normal random variable

The normal random variable is a continuous random variable that has a symmetric density function. The plot of the density function has a bell-shaped form.

A normal random variable is characterised by two parameters, μ and σ^2. Its probability density function is given by

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad x \in \mathbb{R}.$$

Figure 1.2.1 below shows what this density function looks like for multiple values of μ and σ^2. From this figure, it is clear that the density function is symmetric. Thus, the outcome of a normal random variable with parameters μ and σ^2 is with probability $\frac{1}{2}$ smaller than μ, and with probability $\frac{1}{2}$ it is larger. Hence, it is not surprising that the expectation of such a random variable is equal to the parameter μ.

The parameter σ^2 determines how likely it is that the outcome of a normal random variable deviates from its expectation μ. In other words, if σ is large, the bell shape in Figure 1.2.1 will be much wider, and the actual value of the random variable is likely to be further away from μ. Because of this feature, the parameter σ^2 is also called the variance, and the square root of the variance, namely σ itself, is called the standard deviation.

The bell shape of the normal random variable occurs naturally in a variety of settings. Sample averages, for example the average height of a large group of persons, tend to be approximately normally distributed, which is why normal random variables are often encountered.

The distribution function is however hard to compute. If $\mathcal{N}(\mu, \sigma^2)$ denotes a normal random variable with parameters μ and σ^2, we have

$$\mathbb{P}(\mathcal{N}(\mu,\sigma^2) \le t) = \int_{x=0}^{t} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathrm{d}x.$$

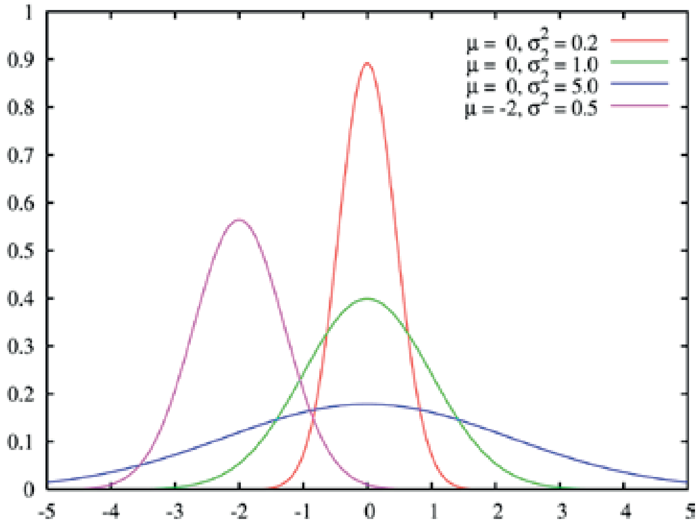

Figure 1.2.1. The density function of a normally distributed random variable with parameters μ and σ^2.

Since there is no easy way to compute this integral, we often use tables such as Table 4.0.2 in Chapter 4, page 50. This table lists, in case $\mu = 0$ and $\sigma = 1$, the value of $\mathbb{P}(\mathcal{N}(0,1) \leq t)$ for various values of t between 0 and 3.4. For example, using the table, we find that

$$\mathbb{P}(\mathcal{N}(0,1) \leq 1.23) = 0.8907,$$

but due to the symmetry also that

$$\begin{aligned}\mathbb{P}(\mathcal{N}(0,1) \leq -0.87) &= \mathbb{P}(\mathcal{N}(0,1) > 0.87) \\ &= 1 - \mathbb{P}(\mathcal{N}(0,1) \leq 0.87) \\ &= 1 - 0.8078 = 0.1922.\end{aligned}$$

But how do we go about finding the distribution function of normal random variables with $\mu \neq 0$ and $\sigma^2 \neq 1$? It turns out that we can then also use this table. For a normal random variable with parameters μ and σ^2, we have for any number u that

$$\mathbb{P}(\mathcal{N}(\mu,\sigma^2) \leq u) = \mathbb{P}\left(\mathcal{N}(0,1) \leq \frac{u-\mu}{\sigma}\right).$$

So, if we want to compute $\mathbb{P}(\mathcal{N}(1,4) \leq 3)$ for example, we use the table and look up the value for $t = \frac{u-\mu}{\sigma} = \frac{3-1}{\sqrt{4}} = 1$, and find that $\mathbb{P}(\mathcal{N}(1,4) \leq 3) = \mathbb{P}(\mathcal{N}(0,1) \leq 1) = 0.8413.$

1.3. Graph theory

An intuitive definition of a network would be a 'collection of objects that are interconnected in some way'. Think for example of a collection of people, who can be interconnected by friendships; or a collection of cities, which can be interconnected by roads. To make this idea precise, we turn to graph theory.

Graph

A **graph** is a pair $G = (V, E)$, where

- V is the set of nodes or vertices;
- E is the set of edges, connecting the nodes.

Typically, we number the nodes from $\{1, 2, 3, \dots, \}$. We denote an edge between two nodes i and j by $\{i, j\}$. To define a graph, we can write down the sets V and E.

EXAMPLE 1.3.1. Consider

$$V = \{1, 2, 3, 4, 5, 6\}, \quad E = \{\{1, 2\}, \{1, 5\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{4, 5\}, \{4, 6\}\}.$$

Then $G = (V, E)$ is a graph with six nodes and seven edges.

It may be very useful to have a graphical representation of a graph. We do this by typically drawing nodes as a circle with a label in it, and edges as a line between nodes. However, you are free to choose any representation you may like! In fact, the location of the nodes is also arbitrary, it only matters the way in which the edges connect the nodes together.

EXAMPLE 1.3.1 (Continued). In Figure 1.3.1 we see two ways in which the graph G can be drawn.

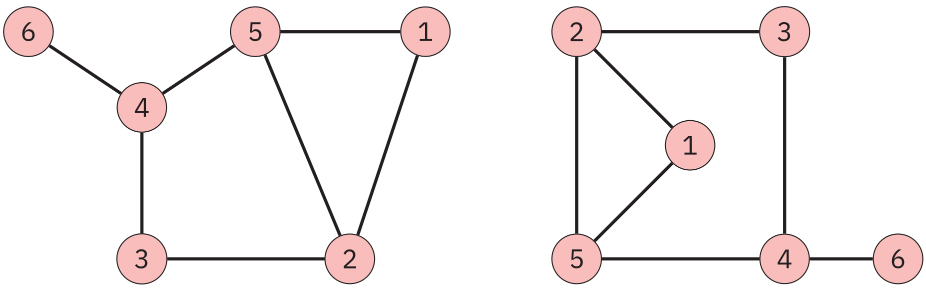

Figure 1.3.1. Two different representations of the graph in Example 1.3.1.

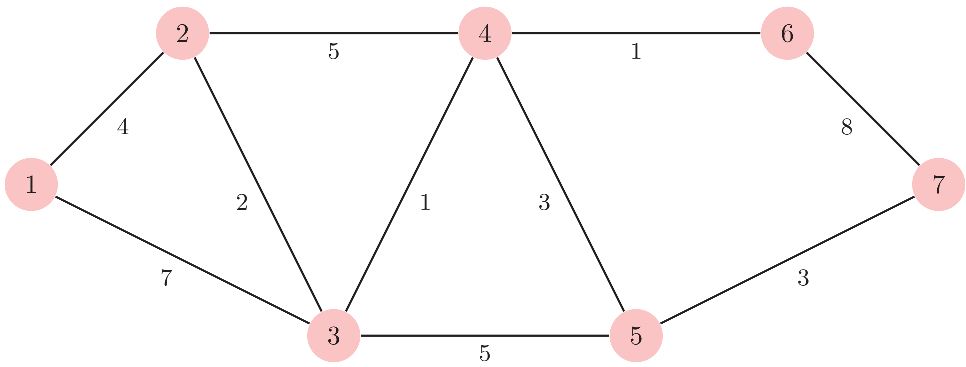

Figure 1.3.2. A network with seven locations

1.3.1. Finding the shortest route within a network

Now that we have seen how graphs can be used to represent networks, there are many questions that can be asked about those networks. For instance, what is the shortest route from one location in a network to another? This is a question that we ask our favourite route planner on a daily basis.

To answer this question, let us look at the network depicted in Figure 1.3.2. This figure contains a graph, and let us assume that each node in this graph is a location in a network. Furthermore, an edge between two of these nodes represents a road between these two locations. The numbers in the figure represent the length of the roads, let us say in kilometres. Using these numbers, we can calculate the distance of a route in a network. For example, in Figure 1.3.2, it is clear that if we would travel from node 1 to node 4 via node 2, the distance traversed would be 4+5 = 9 kilometres. But, if we were to travel from node 1 to node 4 via node 3 instead, we would have to cover 7+1 = 8 kilometres. And if we were to travel from node 1 to node 4 via node 2 and 3, we would have to cover 4+2+1= 7 kilometres. Thus, ‘the shortest route’ from node 1 to node 4 is the route $1 \to 2 \to 3 \to 4$.

For this particular network, we can thus see in an eye blink what the shortest route from node 1 to node 4 is. But the shortest route from node 1 to node 7 is already harder to determine. And then, this is just a network with seven locations. Finding the shortest route in a much larger network by trial and error is simply not doable.

So how do we go about this? We will require a more systematic way of finding the shortest route. Luckily, it exists. We will use an algorithm!

Algorithm

An algorithm is a step-by-step procedure to perform a given task. Algorithms can be executed by computers, but also by persons.

nodes	1	2	3	4	5	6	7
Step 1	-	$(4,1)^*$	$(7,1)$	No edge	No edge	No edge	No edge
Step 2	-	-	$(6,2)^*$	$(9,2)$	No edge	No edge	No edge
Step 3	-	-	-	$(7,3)^*$	$(11,3)$	No edge	No edge
Step 4	-	-	-	-	$(10,4)$	$(8,4)^*$	No edge
Step 5	-	-	-	-	$(10,4)^*$	-	$(16,6)$
Step 6	-	-	-	-	-	-	$(13,5)^*$

Table 1.3.1. Dijkstra's shortest route algorithm for the network in Figure 1.3.2.

Dijkstra's algorithm. More particularly, we will now consider an algorithm that finds the shortest route in a network. This algorithm was conceived in 1959 by Edsger W. Dijkstra, who was a Dutch systems scientist, programmer, software engineer, science essayist and pioneer in computing science.

The algorithm consists of iteratively performing a number of steps. In each of these steps, preliminary routes will be improved and in each step, a definitive shortest route from the starting node to any of the other nodes will be found. For the bookkeeping of these routes, we will keep records on each node. These records give an upper bound on the distance of the shortest route of the starting node to the corresponding node. In each of the steps, these records will be adjusted, and also one of these nodes is marked as 'permanent', indicating that a definitive shortest route from node 1 to the completed node has been found.

All this is perhaps best demonstrated by means of an example: we want to find the shortest route from node 1 to node 7 in Figure 1.3.2. Since the network in this figure has 7 locations, we will need 7-1 = 6 steps of the algorithm. In Table 1.3.1, we will keep track of all the bookkeeping that the algorithm generates.

To start the algorithm, it is worth noting that we already know the shortest route from node 1 to node 1: this route has distance zero, since we are already there! As such, we mark node 1 as 'permanent', and we will not consider node 1 in the steps we are going to perform, as is reflected in the table by '-'. The algorithm can now be started with node 1 as a permanent node. The rest of the nodes are considered 'non-permanent'. In each of the steps, we will check which non-permanent nodes can be reached by permanent ones, and mark a non-permanent node as permanent. We do this as follows.

Step 1: Since node 1 is the only permanent node, we see from Figure 1.3.2 that only nodes 2 and 3 can be reached now from permanent nodes. Node 2 can be reached directly from node 1 with a distance of 4, so in the table, we write $(4,1)$ in the first row (corresponding to Step 1) in the column of node 2. Similarly, we write 7 for node 3 in the first row, as node 3 can be reached directly from node 1 with distance $(7,1)$. In the bracket we always write two numbers, the first number is the distance from a permanent node and the second represents the node from which it can be reached. Nodes 4, 5, 6 and 7 can not be reached directly from node 1, and hence we write 'No

edge' for these nodes in the first row.
The final part of the step consists of marking a non-permanent node as permanent. We will always mark the node with the lowest distance in the row as permanent. In this case, this is node 2 with distance 4 from node 1, so we make node 2 permanent. We denote this by adding an asterisk to the record of node 1 in the first row. The algorithm now says that the shortest route from node 1 to node 2 now simply is the direct route $1 \to 2$ with distance 4.

Step 2: In Step 2 we check whether routes can be made shorter using node 2 as an intermediate node in the route towards other nodes. For node 3, we know from Step 1 that it can be reached directly from node 1 within distance 7. However, since node 2 now is permanent, node 3 can also be reached from node 2: the edge $\{2, 3\}$ has distance 2, and we know that node 2 itself can be reached with distance 4. Therefore, node 3 can also be reached within distance $2 + 4 = 6$, when going via node 2. Therefore, we write $(6, 2)$ for node 3 in Table 1.3.1 in the row corresponding to Step 2 and in the column corresponding to node 3. We conclude that we have made the route from node 1 to node 3 one kilometre shorter!
Node 4 can now also be reached using edge $\{2, 4\}$ with distance 5. As node 2 itself can be reached within distance 4, node 4 can thus now be reached within distance 5+4 = 9 with preceding node 2. Therefore, we write the record $(9, 2)$ in the table.
From nodes 1 and 2, there are still no routes possible to nodes 5,6 and 7, leading to a 'No edge'-record.
Between nodes 3 and 4, node 3 has the shorter distance (namely 6), and therefore we now mark node 3 as permanent with an asterisk.

Step 3: We follow the exact same procedure as the previous steps. Namely, we check whether the now permanent node 3 leads to shorter routes for the other nodes. This is the case for node 4. While in Step 2 we found a distance of 9, we now find a distance 7 via node 3, leading to the record $(7, 3)$. Indeed, node 3 could be reached within distance 6, and the edge $\{3, 4\}$ has distance 1. Node 5 can now be reached via the permanent node 3: namely, node 3 can be reached within distance 6, and edge $\{3, 5\}$ has distance 5, leading to a total distance 6+5=11 and the record $(11, 4)$.
The route to node 4 (distance 7) is now shorter than the route to node 5 (distance 11), so node 4 becomes a permanent node. At this point, nodes 1-4 are permanent, whereas nodes 5, 6 and 7 are still non-permanent. Hence, steps 4-6 will deal with the latter nodes.

Step 4: As node 4 is now a permanent node, we check how this affects the shortest routes of the still non-permanent nodes 5-7. Indeed, the route from node 1 to node 5 can now be made shorter by routing through node 4: we first take the shortest route to node 4 (distance 7 found in step 3) and then use the edge $\{4, 5\}$ (distance 3). This route has distance 7+3=10, which is shorter than the distance 11 in the record for

node 5 in step 3. Therefore, the record for node 5 in step 4 becomes $(10, 4)$. However, node 6 can now also be reached, via the new permanent node 4. This route will have distance 7+1 = 8.
Since node 6 now has the shorter of the two found distances of nodes 5 and 6, node 6 will become permanent. The shortest route from node 1 to node 6 has distance 8, and cannot be made shorter in future steps.

Step 5: There are just two non-permanent nodes left at this point: nodes 5 and 7. There is no direct edge between the most recently permanent node 6 and node 5, so the record of node 5 remains the same as in the previous step: $(10, 4)$. Node 7 can now finally be reached through node 6 (distance 8 found in step 5) and edge $\{6, 7\}$ (distance 8), leading to distance 8+8=16. As a result, we will flag node 5 as permanent, with distance 10 and preceding node 4.

Step 6: Only node 7 is an non-permanent node at this point. We only need to check whether node 5, which we flagged as permanent in the previous step, leads to a shorter route than the one found in step 5 via node 6. This turns out to be the case: if we first go to node 5 (distance 10), and then take the direct edge $\{5, 7\}$ (distance 3), the route will only have distance 13, rather than 16 as found in Step 5. Therefore, the shortest route from node 1 to node 7 has distance with preceding node 5, leading to the record $(13, 5)$ in the table. We finally mark node 7 as permanent, so that there are no non-permanent nodes anymore.

Now that we have performed all the steps of Dijkstra's algorithm, we know that the shortest route from node 1 to node 7 has a distance of 13. To find which route this exactly is, we look at Table 1.3.1, and look at the records with an asterisk (i.e. the records of the nodes when they were marked as a permanent node). In the row of step 6, we see that the preceding node of node 7 is node 5, meaning that the shortest route of node 1 to node 7 coincides with the shortest route of node 1 to node 5, plus the additional edge $\{5, 7\}$. Node 5 was made permanent in step 5 with preceding node 4, meaning that the shortest route from node 1 to node 7 must have the form $1 \to ... \to 4 \to 5 \to 7$. Continuing like this, we find the shortest route $1 \to 2 \to 3 \to 4 \to 5 \to 7$.

Figure 1.3.3. A real time animation of Dijkstra's algorithm on the map of Brielle, the animation can be found on: networkpages.nl/finding-the-shortest-route-to-your-holiday-destination-dijkstras-algorithm/.

On the Network Pages

For further reading on probability theory, algorithms, networks and graph theory have a look at networkpages.nl/category/basic-notions/!

CHAPTER 2

Queueing theory and road traffic analysis

In this chapter some results from queueing theory and road traffic analysis are presented. Section 2.1 is about queueing theory, a branch of probability theory that studies cases and systems where there is demand for some scarce resource. Section 2.2 concerns road traffic analysis and the algorithms that are used in order to determine optimal routes on networks.

2.1. Queueing theory – How long will the queue be?

Queueing theory

Queueing theory is a branch of *operations research* that studies waiting lines or queues from a mathematical perspective.

Some typical everyday examples of queueing systems can be found in supermarkets, industrial production systems and hospitals. In a supermarket customers arrive to the counters, they may have to wait in the queue until their turn comes, they are served and then leave the supermarket. In an industrial production system, like a factory producing cars, the products also have to undergo multiple stages until they are assembled and the servers may be either machines or individuals. Finally, patients arriving to a hospital often need access to resources like doctors, beds, medicine and equipment. A new patient can go into treatment only when the hospital has the necessary resources available, for example only if there are free beds. In short, queueing theory helps us to analyse such systems and make important decisions about the layout, capacity and control.

2.1.1. Queueing model

To study any kind of system or real-life situation we first have to construct a mathematical model. To illustrate what a mathematical model is, you can think of a toy car of a Ferrari (which is also called a *model car*). Such a toy car is not precisely like the Ferrari, since it does not contain a working engine, is made of different material, is much smaller, and so on. However, it does give you a good idea of the shape, how it looks when it is driving, and how it compares to other toy cars. As another example, architects make models of their buildings on a small scale (also called a *scale model*) to study how they would look, how much light will enter the building, how much material is needed, and so on. In a similar way, mathematical models describe a real-life phenomenon, using mathematical concepts and language. The model will not resemble reality perfectly, but can be used to learn from. In our setting we are interested in **queueing models**. The idea behind such a model is to replicate the behavior of a queue as accurately as possible, so that the model can be used to make predictions on how the system will behave. Among others, a queueing model is characterised by:

- **The arrival process of customers.**
  Customers arrive to a system at, possibly random, points in time, we call this the arrival process. The time between two consecutive arrivals is called the interarrival time and is usually described by a random variable. We assume that the interarrival times between customers are independent and have a common probability distribution. In many practical situations customers arrive according to a Poisson stream (i.e.,

the interarrival times have an exponential distribution). Customers may arrive one by one, or in batches. An example of batch arrivals is the customs once at the border where travel documents of bus passengers have to be checked.

- **The behavior of customers.**
  Customers may be patient and willing to wait. Or customers may be impatient and leave after a while. For example, in call centres, customers will hang up when they have to wait too long before an operator is available, and they possibly try again after a while.
- **The service times.**
  Each customer needs some time to be served by the server. This time is called the service time of that customer. Often customers don't have exactly the same service time, hence in many cases we consider the service time to be a random variable. Usually we assume that the service times are independent and have a common distribution function, and that they are independent of the interarrival times. For example, the service times can be deterministic or exponentially distributed. It can also occur that service times depend on the queue length. For example, the processing rates of the machines in a production system can be increased once the number of jobs waiting to be processed becomes too large.
- **The service discipline.**
  Customers can be served one by one or in batches. We have many possibilities for the order in which customers can be served. We mention:
  - first come first served, i.e., in order of arrival;
  - random order;
  - last come first served (e.g., in a computer stack or a shunt buffer in a production line);
  - priorities (e.g., rush orders first, shortest processing time first);
  - processor sharing (in computers that equally divide their processing power over all jobs in the system).
- **The service capacity.**
  There may be a single server or a group of servers helping the customers.
- **The waiting room.**
  There can be limitations with respect to the number of customers in the system, i.e. the customer being served, if any, and the number of customers waiting in the queue. For example, in a data communication network, only finitely many cells can be buffered in a switch. The determination of good buffer sizes is an important issue in the design of these networks.

All these different aspects of queueing systems result in a huge variety of queueing models, which means that an efficient way to characterise queueing models based on its properties is vital. Luckily, D.G. Kendall introduced in 1953 a shorthand notation to characterise queueing models. We explain the notation via the simplest model denoted by $M/M/1$. In the $M/M/1$ queueing model each letter represents a property of the system, in particular:

- **The first letter:** the interarrival time between arriving customers has an exponential distribution with parameter λ. The *M* stands for Memoryless.
- **The second letter:** the service time distribution has an exponential distribution with parameter μ. Other models that are often studied are $M/D/1$ which stands for deterministic service times or $M/G/1$ which stands for general service times.
- **The number:** the number of servers in the queueing model.

2.1.2. The $M/M/1$ queue

After all this notation we can finally start analysing the $M/M/1$ queue. Let L be a random variable that denotes the number of customers in the system. We start analysing the L by constructing a *flow diagram* for L. Below we explain how to construct this flow diagram.

Figure 2.1.1. State space for L.

In this figure the numbers denote the state of the system, i.e., how many customers are in the system. Suppose that $L = i$, that is there are i customers in the system. Then two things can occur, the customer who is being served departs from the system before a new customer arrives, or a new customer arrives before the customer who is being served departs the system. The first event corresponds to the transition $\{L = i\} \to \{L = i - 1\}$ since a customer departs. The service time has an exponential distribution with parameter μ, hence we say that the transition $\{L = i\} \to \{L = i - 1\}$ occurs with rate μ. On the other side the second event corresponds to the transition $\{L = i\} \to \{L = i + 1\}$ since a customer arrives to the system. The interarrival time has an exponential distribution with parameter λ, hence the transition $\{L = i\} \to \{L = i + 1\}$ occurs with rate λ. We can illustrate these transitions using the following flow diagram, where we have chosen the case i = 1.

Figure 2.1.2. Flow from state $\{L = 1\}$.

Doing this for all possible states we obtain the following flow diagram

Figure 2.1.3. Flow diagram for the $M/M/1$ queue.

Since the customers arrive to the system according to a Poisson process, that is at random times, and have a service time which is also random, i.e. exponentially distributed, we observe that L will also be a random variable. Thus we want to know the probabilities that at an arbitrary point in time there will k customers in the system (which means 1 customer in service, and $k-1$ customers waiting for service). We denote this probability by p_k. We are going to compute the probabilities $p_k = \mathbb{P}(L = k)$, for $k = 0, 1, 2, \ldots$ using a *flow conservation argument*.

Flow Conservation Argument

The probability flux in any subset of states is equal to the probability flux out of that subset of states. Intuitively, this means that you enter a state just as many times as you leave a state.

Figure 2.1.4. Flow diagram of the probability flux for the $M/M/1$ queue.

Consider for example the set consisting of the state 0, i.e., where no customers are present in the system. Then the probability flux out of this set is λp_0, because we are in state 0 with probability p_0 and we leave it with rate λ. The probability flux into the set $\{0\}$ is equal to μp_1, because we can reach state 0 only from state 1 in which we are with probability p_1 and the transition from state 1 to 0 happens with rate μ. Then we get the first equation

$$\lambda p_0 = \mu p_1,$$

which we can rewrite to

$$p_1 = \frac{\lambda}{\mu} p_0 = \rho p_0, \tag{2.1.1}$$

where $\rho = \lambda/\mu$. If $\rho < 1$, then ρ is called the *occupation rate*, because it is the fraction of time the server is working. Intuitively, $\rho < 1$ means that there are on average more departures than arrivals so the queue will not keep growing. Suppose now that we consider the subset $\{1\}$, then we obtain the equation

$$(\lambda + \mu)p_1 = \lambda p_0 + \mu p_2,$$

which, after substituting p_1 from (2.1.1), can be rewritten to

$$p_2 = \rho^2 p_0.$$

In general we obtain the equations

$$(\lambda + \mu)p_k = \lambda p_{k-1} + \mu p_{k+1}, \qquad k = 1, 2, \ldots$$

and

$$p_k = \rho^k p_0, \qquad k = 0, 1, 2, \ldots.$$

Hence it suffices to compute p_0, which denotes the probability that there are no customers waiting, and there is nobody being served. We know that the sum of all the probabilities has to be equal to one, hence

$$\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} \rho^k p_0 = 1.$$

Solving this equation yields

$$p_0 = 1 - \rho.$$

For derivation of this result have a look at Exercise 3.2.1. Hence we obtain the following result for the desired probabilities

$$p_k = \rho^k (1 - \rho), \quad k = 0, 1, 2, \ldots$$

Hence, the number of customers in an M/M/1 system is a geometric random variable with success probability $1 - \rho$ (see Section 1.2.3). With this result already some quantities can be computed. For example the average number of customers in the system is equal to

$$\mathbb{E}[L] = \sum_{k=0}^{\infty} k p_k = \sum_{k=0}^{\infty} k \rho^k (1 - \rho) = \frac{\rho}{1 - \rho}. \qquad (2.1.2)$$

See also Exercise 3.2.1 for more details on how to derive this result.

2.1.3. Mean value approach

The analysis performed in the previous section is only valid for exponential service times, or else only in the $M/M/1$ model. In this section we will present a technique called the *mean value approach*. This approach can be used to analyse queueing models with general service times, that is the $M/G/1$ queue. A short reminder, in the $M/G/1$ queue customers arrive according to a Poisson process to the system and their service time has a general probability distribution, i.e. it is not necessarily the exponential distribution. Let's see how the mean value approach works.

Consider a customer who just arrived at the queue. There are two possibilities, either the server is busy with another customer or the server is free. If the server is busy with another customer then an arriving customer needs to wait for some additional time until this customer is fully served. The customer in service has been in service for some time, hence the remaining service time is not equal to the full service time. This remaining service time is called the *residual service time* of the customer in service. Next to the customer in service there may also be customers in the queue who arrived earlier. An arriving customer has to wait for all the customers who have been already waiting in the queue, if any. If there are no customers in the system and the server is free then the arriving customer does not have to wait at all. Let's quantify this argument in a formula giving the average waiting time of an arriving customer at the system.

Waiting Time

The waiting time of an arriving customer, denoted by W, is a random variable representing the time an arriving customer has to wait in the queue. If the system is empty then the waiting time is zero.

The service times of customers are random variables which are independent of each other and have the same probability distribution. Let the random variable B denote the service time, let R denote the residual service time of a customer in service and let L^Q denote the number of customers waiting in the queue. Then,

$$\mathbb{E}[W] = \mathbb{E}[L^Q]\mathbb{E}[B] + \rho\mathbb{E}[R]. \tag{2.1.3}$$

Here $\mathbb{E}[L^Q]\mathbb{E}[B]$ is the expected time the arriving customers has to wait for customers in the queue and $\rho\mathbb{E}[R]$ is the expected time the arriving customer has to wait for the customer in service. The argument above and the equation in 2.1.3 show how the mean value approach can be applied to study queueing systems. The quantities $\mathbb{E}[B]$ and $\mathbb{E}[R]$ are in general known when the distribution of the service time is given.

Average Residual Service Time

The average residual service time, denoted by $\mathbb{E}[R]$, is given by the formula

$$\mathbb{E}[R] = \frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}, \tag{2.1.4}$$

where B denotes the service time and

$$\mathbb{E}[B^2] = \int_0^\infty x^2 f_B(x)\mathrm{d}x.$$

In the equation above f_B denotes the probability density function of the random variable B.

The exact derivation of the equation in (2.1.4) is above the scope of this booklet and it thus omitted. Returning to (2.1.3), we observe that this equation contains two unknown quantities, namely $\mathbb{E}[W]$ and $\mathbb{E}[L^Q]$. Hence in order to be able to find $\mathbb{E}[W]$ and $\mathbb{E}[L^Q]$ we need one more equation. This equation is given by Little's law!

Little's law Little's law is the most important relation between $\mathbb{E}[L^Q]$, the mean number of customers in the queue, $\mathbb{E}[W]$, the mean waiting time of a customer and λ, the average number of customers entering the system.

Little's Law

Little's law states that

$$\mathbb{E}[L^Q] = \lambda\mathbb{E}[W].$$

Intuitively, this result can be understood as follows. Suppose that all customers pay 1 euro per unit time while in the queue. This money can be earned in two ways.

- The first possibility is to let pay all customers continuously in time. Then the average reward earned by the system equals $\mathbb{E}[L^Q]$ euro per unit time.
- The second possibility is to let customers pay 1 euro per unit time for their residence in the queue when they leave. In equilibrium, the average number of customers leaving the system per unit time is equal to the average number of customers entering the system. So the system earns an average reward of $\lambda\mathbb{E}[W]$ euro per unit time.

The system earns the same in both cases. Now by substituting this in Equation 2.1.3 we find

$$\mathbb{E}[W] = \frac{\rho\mathbb{E}[R]}{1-\rho}, \tag{2.1.5}$$

where $\rho = \frac{\lambda}{\mu}$.

EXAMPLE 2.1.1. Suppose that the service time is exponentially distributed with parameter μ. Then

$$\mathbb{E}[B] = \frac{1}{\mu} \qquad \text{and} \qquad \mathbb{E}[B^2] = \frac{2}{\mu^2}. \tag{2.1.6}$$

From (2.1.4) we obtain

$$\mathbb{E}[R] = \frac{\frac{2}{\mu^2}}{2\frac{1}{\mu}} = \frac{1}{\mu}.$$

Hence from (2.1.5) we obtain

$$\mathbb{E}[W] = \frac{\rho}{1-\rho}\frac{1}{\mu}.$$

In Exercise 3.2.1(3) you are asked to verify the result in (2.1.2) for $\mathbb{E}[L]$.

2.1.4. Conservation law

So far we only considered the first-come-first-served service discipline (FCFS). Now, we will extend this. Consider a single-server queue with r types of customers. Type i customers arrive according to a Poisson arrival stream with rate λ_i, $i = 1, \dots, r$. The mean service time and mean residual service time of a type i customer is denoted by $\mathbb{E}[B_i]$ and $\mathbb{E}[R_i]$. Customers enter service in an order independent of their service times and they may not be interrupted during their service. So, for example, the customers may be served according to FCFS, random order or a non-preemptive priority rule. Below we derive a conservation law for the mean waiting times of the r type of customers, which expresses that a weighted sum of these mean waiting times is independent of the service discipline. This implies that an improvement in the mean waiting time of one customer type owing to a service discipline will always degrade the mean waiting time of another customer type.

Let $\mathbb{E}[V(P)]$ and $\mathbb{E}[L_i^Q(P)]$ denote the mean amount of work in the system and the mean number of type i customers waiting in the queue, respectively, for service discipline P. The mean amount of work in the system is given by

$$\mathbb{E}[V(P)] = \sum_{i=1}^{r} \mathbb{E}[L_i^Q(P)]\mathbb{E}[B_i] + \sum_{i=1}^{r} \rho_i \mathbb{E}[R_i].$$

Clearly the residual service time does not depend on the discipline P. The crucial observation is that the amount of work in the system does not depend on the order in which the customers are served. The amount of work decreases with one unit per time unit independent of the customer being served and when a new customer arrives the amount of work is increased by the service time of the new customer. Hence, the amount of work does not depend on P. Using Little's law

$$\mathbb{E}[L_i^Q(P)] = \lambda_i \mathbb{E}[W_i(P)], \qquad \text{for all } i = 1, \dots r.$$

Hence we obtain the following conservation law for the mean waiting times,

$$\sum_{i=1}^{r} \rho_i \mathbb{E}[W_i(P)] = \text{constant with respect to the service discipline } P.$$

Figure 2.1.5. Consult a mathematician before you visit Disneyland, because queues can be large! By Ellen Cardinaels.

On the Network Pages

For further reading on queueing theory and its applications have a look at:

(1) *The quest for a beter Internet* by Mark van der Boor,
networkpages.nl/the-quest-for-a-better-internet/.

(2) *Consult a mathematician before you visit Disneyland* by Ellen Cardinaels,
networkpages.nl/consult-a-mathematician-before-you-visit-disneyland/.

(3) *Can flipping the queue spare you time* by Youri Raaijmakers,
networkpages.nl/can-flipping-the-queue-spare-you-time/.

(4) *Traffic lights no longer needed: back to the future* by Rik Timmerman,
networkpages.nl/traffic-lights-no-longer-needed-back-to-the-future/.

Acknowledgements

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2.2. Road traffic analysis – Route selection in a network

A road traffic network consists of a collection of roads that connects various cities. The users of this network want to travel from their current location, called the origin, to some desired destination. These users travel in a vehicle and will therefore be called the drivers in the network.
It will often be the case that a driver is able to choose between different routes to move from their origin to their destination. The routes they can choose from will have different characteristics. For example, when entering a destination in a navigation system, the device will ask the user to choose between the fastest route and the shortest route. It will depend on the preferences of the driver which route is chosen. A driver with ample time who is concerned about the fuel consumption of his car may select the shortest route, whereas someone who is in a hurry is likely to choose the fastest route.
It is not difficult to determine the length of a route that connects the origin and the destination. The length of the individual road segments that make up the route are fixed, and the total travel distance is simply the sum of these segments. We can therefore say that the distance between origin and destination is a deterministic quantity. The travel time between origin and destination, however, is certainly not deterministic. For example, you may get stuck in a traffic jam as a result of an accident which will drastically increase your travel time. One could also think of smaller hindrances, such as having to wait for a bridge that is opening or repeatedly having to stop for red lights. Some routes will be more likely to cause delays than others. These routes are said to bear a higher risk related to the travel time. Therefore, in contrast to travel distances, travel times are stochastic.
Since travel times are stochastic, it is important to realise that the fastest route suggested by your navigation system is only the fastest route in expectation. It may be the case that this route consists of roads that are likely to cause delays. As a result, the actual travel time of this route may be very uncertain. It is because of this uncertainty that the fastest route is not always the most desirable route.

2.2.1. Network representation

Let's try to present the previous discussion as a network. Since many roads are two-way streets, it is natural to express a road traffic network as an undirected graph $G = (V, E)$. In Figure 2.2.1 below we illustrate the graph G with

$$V = \{1, 2, 3, 4, 5, 6, 7\},$$
$$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{4, 6\}, \{5, 7\}, \{6, 7\}\}.$$

The vertices represent cities and the edges represent roads that link these cities. Given an origin and a destination in this road traffic network, we are already able to determine

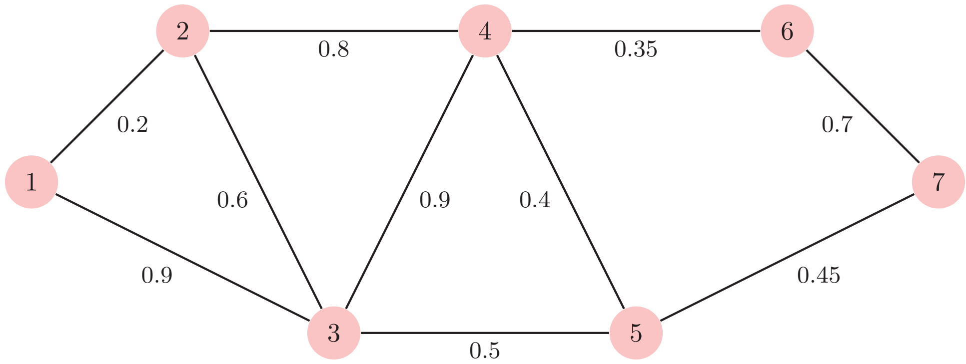


Figure 2.2.1. Most reliable route network model

the shortest route using Dijkstra's algorithm. With slight adjustments, however, we may use Dijkstra's algorithm to find routes that minimise other characteristics. For example, consider a driver that is not interested in finding the shortest route, but who wants to find the route that minimises the probability of getting stuck in a traffic jam. Note that this route may be a big detour from the shortest route and is therefore unlikely to be the fastest route. We call this route the most reliable route. The following is based on Example 6.3-2 in the book "*Operations Research, An Introduction*" (9th Edition) by Hamdy A. Taha.
In Figure 2.2.1 we assigned to each edge $\{i, j\}$ a probability p_{ij} of not running into a traffic jam on the road between city i and j. For example, $p_{12} = 0.2$ and $p_{35} = 0.5$. Note that there is no direct link between city 1 and 7, but the route $1 \to 2 \to 4 \to 6 \to 7$ is a possible route between those cities and the probability of not running into a traffic jam on this route is

$$p_{17} = p_{12} \times p_{24} \times p_{46} \times p_{67} = 0.2 \times 0.8 \times 0.35 \times 0.7 \approx 0.04.$$

Hence, if the driver chooses this route, there is a probability of not running into a traffic jam of only 4%. This does not look very promising indeed! Perhaps we are able to find a route on which it is less likely to get stuck in traffic?
This problem can be formulated as a shortest route model by using a logarithmic transformation. This way, we can convert the product of probabilities into a sum of logarithms of probabilities. That is, the probability assigned to our previously suggested route is transformed to

$$p_{17} = p_{12} \times p_{24} \times p_{46} \times p_{67} \implies \log p_{17} = \log p_{12} + \log p_{24} + \log p_{46} + \log p_{67}.$$

If we are able to find a route that maximises $\log p_{17}$, this same route would also maximise the actual probability p_{17} of not running into a traffic jam. This is due to the fact that the logarithm is a strictly increasing function. Note that Dijkstra's algorithm is designed to find a route that minimises a sum instead of maximising it. However, this problem can easily be

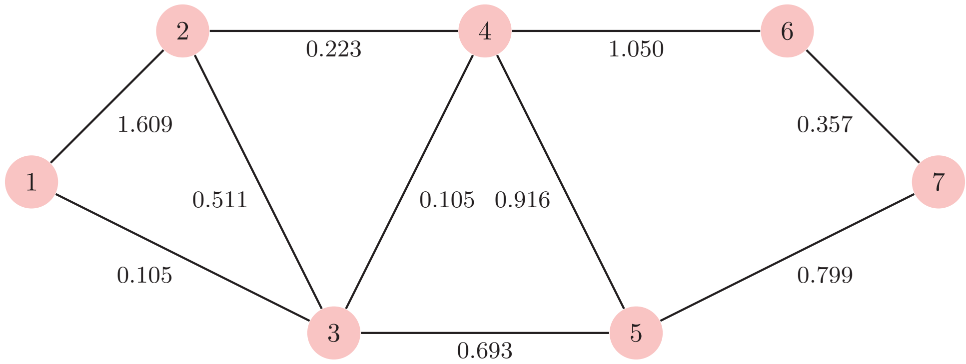


Figure 2.2.2. Most reliable route representation as a shortest route model

countered by minimising $-\log p_{17}$. In Figure 2.2.2 we have replaced each p_{ij} by $-\log p_{ij}$. We have now successfully converted our problem to the shortest route problem which we looked at in Section 1.3.1, since the shortest route of the network in Figure 2.2.2 corresponds to the most reliable route in the sense that this route has the highest probability of not running into a traffic jam.

To find the most reliable route, we thus use the weights in Figure 2.2.2 and act as if they are distances. Making a table for the weights using Dijkstra's algorithm, in the same way as we did in Section 1.3.1, leads to Table 2.2.1.

nodes	2	3	4	5	6	7
Step 1	$(1.609, 1)$	$(0.105, 1)^*$	No edge	No edge	No edge	No edge
Step 2	$(0.616, 3)$	-	$(0.210, 3)^*$	$(0.798, 3)$	No edge	No edge
Step 3	$(0.433, 4)^*$	-	-	$(0.798, 3)$	$(1.260, 4)$	No edge
Step 4	-	-	-	$(0.798, 3)^*$	$(1.260, 4)$	No edge
Step 5	-	-	-	-	$(1.260, 4)^*$	$(1.597, 5)$
Step 6	-	-	-	-	-	$(1.597, 5)^*$

Table 2.2.1. Dijkstra's shortest route algorithm for the network in Figure 2.2.2.

From this table, we see that the most reliable route from city 1 to city 7 is the route $1 \to 3 \to 5 \to 7$. As a side note, if we would like to go from city 1 to city 2 instead, the most reliable route is not the direct route $1 \to 2$, but by following the records in the table, the most reliable route in this case would be $1 \to 3 \to 4 \to 2$. When looking at Figure 2.2.1, this seems right: the direct probability 0.2 of not running into a traffic jam is indeed smaller than the probability $0.9 \times 0.5 \times 0.45 = 0.2025$ of the route found by Dijkstra's algorithm!

2.2.2. Refinements using probability distributions

So far, we have seen that we can use Dijkstra's algorithm in order to both find the shortest and the most reliable route. However, these routes may both be undesirable for a user of the road traffic network. The shortest route may be a very slow route due to congestion, whereas the most reliable route may be a big detour from the destination, resulting in a long travel time.

A much more realistic objective that a driver may have is that they want to arrive at their destination 'on time'. For example, consider the situation where a driver has a meeting in one hour and they want to maximise the probability of not being late. Perhaps the most reliable route is the best choice, as the low uncertainty ensures that the driver will arrive on time. Another extreme is the situation in which a high risk route, meaning a short route which is likely to be congested, is the only route which gives a chance of arriving on time. Of course, in this case the high risk route would be the best route. But how do we determine the best route mathematically?

Instead of looking at the length or the reliability of a road between two cities, a more relevant quantity would be the travel time between these two cities. We have already argued before that the travel time is a stochastic quantity. This is where probability distributions come in. Routes that are expected to have a short travel time will have a probability distribution that is centred around a relatively low value. Unreliable routes, i.e. routes on which there is high uncertainty over the travel time, will have a probability distribution that is more spread.

Let's go back to the situation in Figure 2.2.1. The edge between city 1 and 3 indicates that there is a low probability of getting stuck in a traffic jam on this road. Therefore, the travel time to move between city 1 and 3 will be fairly certain and its probability distribution will be relatively concentrated. In contrast, the road between city 1 and 2 is likely to be congested making the travel time between these two cities rather uncertain. Hence, this probability distribution will be more spread, or equivalently, the travel time will have a higher variance.

For simplicity, we will assume that the travel times are distributed according to a normal distribution. An objection to this assumption is that the normal distribution also assigns positive probability to negative values, which does not make sense in our application. However, this distribution is intuitive to work with and it allows us to do some explicit computations without the use of a computer.

In Figure 2.2.3 we once again consider the same road traffic network, but we are now concerned with the random travel times between the cities instead of the probability of getting stuck in a traffic jam. We let the random variable T_{ij} denote the travel time between city i and j and we write $T_{ij} \sim \mathcal{N}(\mu_{ij}, \sigma_{ij}^2)$ to indicate that T_{ij} is normally distributed with mean μ_{ij} and variance σ_{ij}^2. Possible densities for the travel time distribution between city 1 and 3 and city 1 and 2 that would agree with the previous discussion are given in Figure 2.2.4.

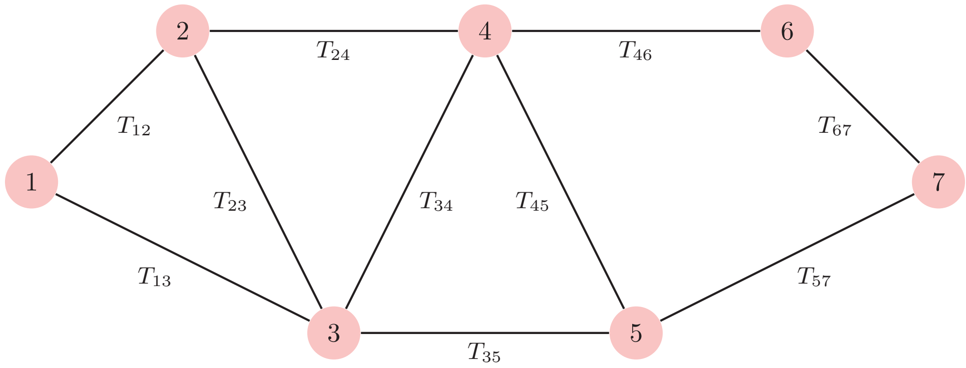


Figure 2.2.3. Road traffic model with stochastic travel times

What is interesting to note here is that even though $13 = \mu_{12} < \mu_{13} = 15$, inspection of these densities seems to imply that

$$\mathbb{P}(T_{12} \geq 20) > \mathbb{P}(T_{13} \geq 20). \tag{2.2.1}$$

It is remarkable that the road between city 1 and 2 has a lower mean travel time compared to the route between city 1 and 3, but at the same time this route is more likely to take more than 20 minutes to traverse. This is of course caused by the higher variance of T_{12}. This example demonstrates that the expected fastest route, which is the route that is most likely to be suggested by a navigation system, is not necessarily the best route for drivers that want to maximise the probability of arriving at their destination on time. If we want to determine the best route for these drivers, it is crucial that we know the probability distribution of the travel times.

2.2.3. Concluding remarks

We now know that the road traffic network in Figure 2.2.1 does not tell the whole story. Instead of only knowing the expected travel times and the reliability of a road network, knowing the actual probability distribution of the travel times would provide us with much more information. This in turn allows us to answer questions that are of greater relevance for some drivers.

The model we discussed in which the travel times are assumed to be independent and normally distributed could already be fruitful in practice in order to find routes that are both fast and reliable. However, there is room for improvement as the assumptions we made are not very realistic.

For one, we have already argued that travel times cannot be normally distributed. This is due to the normal distribution assigning positive probability to negative values. In other words, if travel times are normally distributed, there is a positive probability of having a

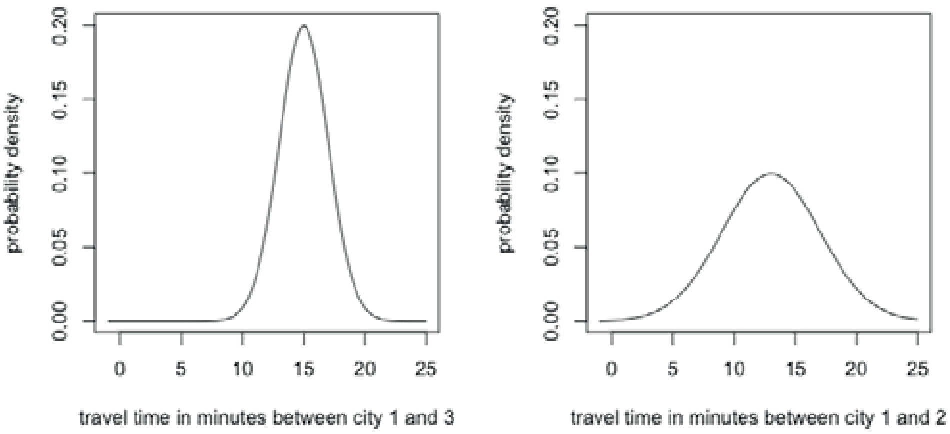


Figure 2.2.4. The density of $T_{13} \sim \mathcal{N}(15, 4)$ *(left) and* $T_{12} \sim \mathcal{N}(12, 16)$ *(right).*

negative travel time. Nonsense! Therefore, it is better to assume that the travel times follow some non-negative distribution such as the log-normal distribution or the gamma distribution. Even worse, it can happen that none of the well-known probability distributions provide a good explanation of the actual travel times. In this case we would have to resort to so-called non-parametric methods. Another assumption we made is that the travel times are independent across roads. This means that any information regarding the travel time of one road has no impact on the travel time of any other road. However, one could argue that the level of congestion of a road is positively correlated with the level of congestion of the adjacent roads. This violates the assumption, since the level of congestion clearly has an impact on the travel times. These issues greatly complicate the analysis of the network.

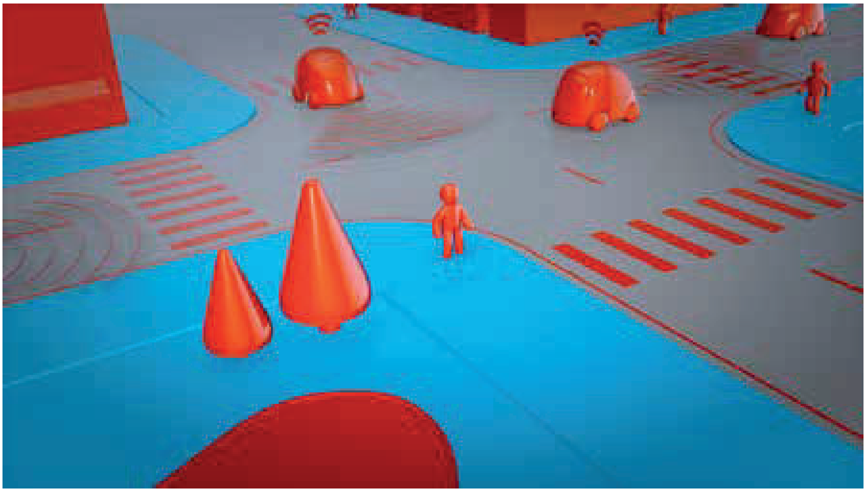


Figure 2.2.5. Reducing travel times can be achieved in multiple ways, finding the optimal route is one of them as we discussed. But there is more! You can also monitor traffic, that is what traffic lights do for example. networkpages.nl/traffic-lights-no-longer-needed-back-to-the-future/

On the Network Pages

For more information on algorithms, networks and road traffic analysis have a look at:

(1) *Finding the shortest route to your holiday destination: Dijkstra's algorithm* by Bart Jansen,

networkpages.nl/finding-the-shortest-route-to-your-holiday-destination-iv-dijkstra-algorithm/.

(2) *How to plan Valentine's day using a matching algorithm* by Bart Jansen,

networkpages.nl/how-to-plan-valentines-day-using-a-matching-algorithm/.

(3) *Can Traffic Congestion: Braess' Paradox* by Peter Kleer,

networkpages.nl/traffic-congestion-iv-braess-paradox/.

(4) *Traffic lights no longer needed: back to the future* by Rik Timmerman,

networkpages.nl/traffic-lights-no-longer-needed-back-to-the-future/.

CHAPTER 3

Exercises

This chapter contains exercises on the theory presented in chapter 1 and 2.

3.1. Exercises on probability theory

3.1.1. Conditional probabilities and expectations

A conditional probability is denoted by $\mathbb{P}(A|B)$, which corresponds to

the probability of A happening, given that B happens.

Let's look at a few simple examples. We denote by X the random variable that represents the number that you roll with a six-sided die.

(1) What is the probability that you roll a 6 with a six-sided die? In a formula: $\mathbb{P}(X = 6)$.
(2) What is the probability that you roll a 6, given that you roll at least a 4; $\mathbb{P}(X = 6|X \geq 4)$?
(3) You can use the following formula to compute conditional probabilities:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}. \tag{3.1.1}$$

Check that this formula works by solving the second question again, but now with the formula.
(4) Similarly to probabilities, we can also look at expectations. What is the expected number you roll with a six-sided die? In formulas: $\mathbb{E}[X]$.
(5) What is the expected number that you roll, given that you roll at least a 4; $\mathbb{E}[X|X \geq 4]$?

3.1.2. The exponential distribution

The exponential distribution is defined in the following way. Suppose that X is exponentially distributed with parameter λ. Then $\mathbb{P}(X < t) = 1 - e^{-\lambda t}$.

(1) Calculate $\mathbb{P}(X \geq t)$.
(2) Calculate $\mathbb{P}(1 < X < 2)$.
(3) Calculate the expectation of the exponential distribution with the following formula:

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq t)dt.$$

(4) Use Equation (3.1.1) to prove the memoryless property of the exponential distribution:

$$\mathbb{P}(X > t + u|X > t) = \mathbb{P}(X > u).$$

3.2. Exercises on queueing theory

3.2.1. Mean queue length

We introduce $\rho = \lambda/\mu$ to make the calculations easier. In the $M|M|1$ queue we found that the probability of having k customers in the system, in equilibrium, equals

$$p_k = (1-\rho)\rho^k, \qquad k = 0, 1, 2, \ldots.$$

(1) Of course, the sum of all these probabilities should sum up to 1. Prove that $\sum_{i=0}^{\infty} p_i = 1$.
(2) We can calculate the mean queue length using these probabilities;

$$\mathbb{E}[L] = \sum_{k=0}^{\infty} k p_k = \sum_{k=0}^{\infty} k(1-\rho)\rho^k.$$

Calculate $\mathbb{E}[L]$.
(3) In Example 2.1.1 we derived an expression for the mean waiting time $\mathbb{E}[W]$. Use Little's law to find $\mathbb{E}[L^Q]$. What is the relation between $\mathbb{E}[L]$ and $\mathbb{E}[L^Q]$? Use this relation to compute $\mathbb{E}[L]$.

3.2.2. Busy supermarket

Think of a supermarket in your area. We model the checkout of this supermarket as an $M/M/1$ queue with an arrival rate of 60 customers per hour, thus $\lambda = 1$ (customer per minute), and a mean service team of 45 seconds, thus $\mu = 1.33$ (customer per minute).

(1) Draw the flow diagram for this $M/M/1$ queue.
(2) Determine the probabilities $\mathbb{P}(L = 0)$, $\mathbb{P}(L = 1)$ and $\mathbb{P}(L = 2)$.

We call the supermarket crowded when there are 3 or more customers in the system.

(3) On average, what is the fraction of the day that the this supermarket is crowded?

3.2.3. Broken televisions

A repair man fixes broken televisions. The repair time is exponentially distribution with a mean of 0.5 hour, thus $\mu = 2$. Broken televisions arrive at his repair shop according to a Poisson stream, on average 10 broken televisions per day (8 hours), thus $\lambda = \frac{8}{10} = 0.8$.

(1) What is the fraction of time that the repair man has no work to do?
(2) How many televisions are, on average, at his repair shop?
(3) What is the mean sojourn time (waiting time plus repair time) of a television?

Hint to (3)

From Little's law we had the equation

$$\mathbb{E}[L^Q] = \lambda \mathbb{E}[W],$$

where L^Q denoted the number of jobs waiting in the queue and W was the time a job has to wait in the queue before starting being served. Derive a similar expression for L (the number of jobs in the system, queue and in service) and S (sojourn time) instead of L^Q and W.

Instead of an exponentially distributed repair time, now the repair time is deterministic equal to 0.5.

(4) What is now the mean sojourn time of a television? Compare your answer to your answer at c).

3.2.4. Extension of the single-server queue

Previously we drew the transition diagram and calculated the equilibrium probabilities of the $M|M|1$ queue, which is a system where 1 job can be served at a time. In this set of questions, we will consider three extensions.

(1) The $M|M|c$ queue is an extension of this model, where up to c jobs can get service simultaneously. Draw the transition diagram of the $M|M|c$ queue.

Hint

Suppose two jobs are getting service at the same time. The rate at which servers move from having 2 to 1 jobs, is equal to $2 \cdot \mu$.

Calculate the equilibrium probabilities of the $M|M|c$ queue.

(2) In the $M|M|1|k$ queue, only one job receives service at a time. The k in the name denotes that there are finitely many spots to wait in the queue. At any moment, there can be at most k jobs in this system. Whenever a job arrives and the system is full, it will be blocked and it will leave forever. Draw the transition diagram, calculate the equilibrium probabilities, and find the blocking probability; the probability that an arbitrary job will be blocked.

(3) The $M|M|c|k$ model is a mix of the $M|M|1|k$ and the $M|M|c$. In this system there are c servers, hence c jobs can receive service simultaneously, and at most k jobs can reside in the queue. Can you find the transition diagram, equilibrium probabilities and blocking probability?

3.3. Exercises on road traffic analysis

3.3.1. Applying Dijkstra's algorithm

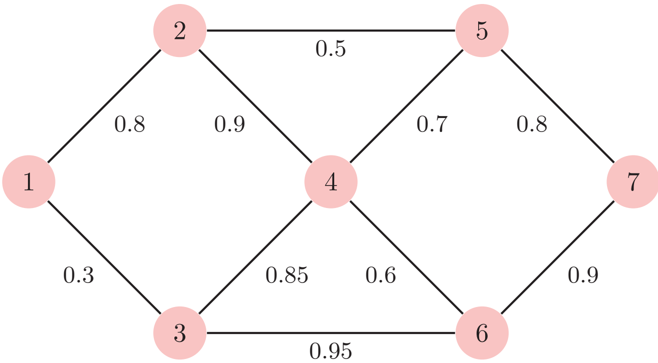


Figure 3.3.1. Road traffic network for Exercise 3.3.1

Figure 3.3.1 shows the possible routes to move from city 1 to city 7, and the associated probabilities of not running into a traffic jam. Use Dijkstra's algorithm to find the most reliable route of this road traffic network. What is the probability of not getting stuck in a traffic jam?

3.3.2. Normal Distribution

Verify the claim in (2.2.1). Recall that $T_{12} \sim \mathcal{N}(12, 16)$ and $T_{13} \sim \mathcal{N}(15, 4)$. Use Table 4.0.2 on page 50.

3.3.3. Road Traffic Analysis

(1) Let X and Y be continuous random variables in $\mathbb{R}$. Show that

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

You may use that $\int_{\mathbb{R}} f(x, y)\mathrm{d}y = f(x)$ and $\int_{\mathbb{R}} f(x, y)\mathrm{d}x = f(y)$.

(2) Assume that the travel time distributions of the road traffic network in Figure 2.2.3 are known and are given in Table 3.3.1 below. Find the route between city 1 and 7 that has the lowest expected travel time.

(3) Suppose you have a job interview in 40 minutes. Find the probability that you arrive on time if you take the route you found in part (2). What is this probability if you take the route $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$? What do you observe?

	T_{12}	T_{13}	T_{23}	T_{24}	T_{34}	T_{35}	T_{45}	T_{46}	T_{57}	T_{67}
μ	12	10	2	7	4	4	3	10	19	5
σ^2	1	9	1	9	4	1	1	16	1	4

Table 3.3.1. T_{ij} denotes the travel time between city i and j in minutes and is normally distributed with parameters μ and σ^2.

Hint

Recall that the travel times are independent across the different roads. You can use that if $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ are independent, it holds that $X + Y \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

CHAPTER 4

Solutions to the exercises

This chapter contains the solutions to some of the exercises in chapter 3.

4.1. Probability theory

4.1.1. Conditional probabilities and expectations

(1)

$$\mathbb{P}(X=6)=\mathbb{P}(\text{you get a 6 when rolling a six-sided die})=\frac{1}{6},$$

since it is equally probable to obtain any of the six sides.

(2) This is a conditional probability. You don't know exactly what the outcome is but you know that it is at least 4. This means that the die number is either a 4 or a 5 or a 6. Yes now you have three possible outcomes, given the condition, not six. All three are equally probable, hence the desired probability is equal to

$$\mathbb{P}(X=6|X\geq 4)=\frac{1}{3}.$$

(3)

$$\mathbb{P}(X=6|X\geq 4)=\frac{\mathbb{P}(\{X=6\}\text{ and }\{X\geq 4\})}{\mathbb{P}(X\geq 4)}=\frac{\frac{1}{6}}{\frac{1}{2}}=\frac{1}{3}. \tag{4.1.1}$$

(4)

$$\mathbb{E}[X]=\sum_{i=1}^{6}i\mathbb{P}(X=i)=\frac{1}{6}\sum_{i=1}^{6}=3.$$

(5)

$$\mathbb{E}[]X|X\geq 4]=\sum_{i=1}^{6}i\mathbb{P}(X=i|X\geq 4)=5.$$

4.1.2. The exponential distribution

(1)

$$\mathbb{P}(X\geq t)=1-\mathbb{P}(X<t)=e^{-\lambda t}.$$

(2)

$$\mathbb{P}(1<X<2)=\mathbb{P}(X<2)-\mathbb{P}(X<1)=e^{-\lambda}-e^{-2\lambda}.$$

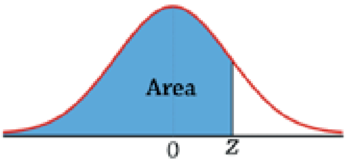
(3)

$$\mathbb{E}[X]=\frac{1}{\lambda}.$$

(4)

$$\mathbb{P}(X>t+u\text{ and }X>t)=\mathbb{P}(X>t+u),$$

because if $X>t+u$ then it will also happen that $X>t$. The rest follows by doing one more computation.



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Figure 4.0.2. Values of distribution function of normal distribution

4.2. Queueing theory

4.2.1. Mean queue length

(1)

$$\sum_{i=0}^{\infty} p_i = \sum_{i=0}^{\infty} (1-\rho)\rho^i = (1-\rho)\sum_{i=0}^{\infty} \rho^i.$$

Geometric sum

For the geometric sum we have that

$$\sum_{i=0}^{n} \omega^i = \frac{1-\omega^{n+1}}{1-\omega}.$$

Hence we have that

$$\sum_{i=0}^{\infty} \omega^i = \lim_{n\to\infty} \sum_{i=0}^{n} \omega^i = \lim_{n\to\infty} \left(\frac{1-\omega^{n+1}}{1-\omega} \right),$$

and hence for $\omega \in (0,1)$

$$\sum_{i=0}^{\infty} \omega^i = \frac{1}{1-\omega}.$$

Using this result the answer follows.

(2)

$$\begin{aligned}\sum_{i=0}^{\infty} i(1-\rho)\rho^i &= \sum_{i=1}^{\infty} i(1-\rho)\rho^i = \rho(1-\rho)\sum_{i=1}^{\infty} i\rho^{i-1} \\ &= \rho(1-\rho)\left(\sum_{i=0}^{\infty} \rho^i\right)' = \rho(1-\rho)\left(\frac{1}{1-\rho}\right)' = \frac{\rho}{1-\rho}.\end{aligned}$$

(3) Applying Little's law we compute

$$\mathbb{E}[L^Q] = \frac{\rho^2}{1-\rho}.$$

The random variable L denotes the number of customers in the system, which is equal to the number of customers in the queue, i.e. L^Q, plus one if there is a customer being served. Hence

$$L = L^Q + 1_{\{\text{Customer in service}\}},$$

where $1_{\{\text{Customer in service}\}}$ is equal to 1 if there is a customer being served, which happens with probability ρ, and is equal to 0 otherwise. Hence

$$\mathbb{E}[L] = \mathbb{E}[L^Q] + \rho = \frac{\rho^2}{1-\rho} + \rho = \frac{\rho}{1-\rho}.$$

4.2.2. Busy supermarket

(1) The flow diagram is

(2)

$$\begin{aligned}
\mathbb{P}(L=0) &= p_0 = \rho^0(1-\rho) = (1-\rho) = (1-\frac{1}{1.33}) = 0.25 \\
\mathbb{P}(L=1) &= p_1 = \rho^1(1-\rho) = \frac{1}{1.33}(1-\frac{1}{1.33}) = 0.187 \\
\mathbb{P}(L=2) &= p_2 = \rho^2(1-\rho) = (\frac{1}{1.33})^2(1-\frac{1}{1.33}) = 0.140
\end{aligned}$$

(3)

$$1 - \mathbb{P}(L=2) - \mathbb{P}(L=1) - \mathbb{P}(L=0) = 1 - 0.25 - 0.187 - 0.140 = 0.423$$

4.2.3. Broken televisions

(1)

$$1 - \rho = 1 - \frac{0.8}{2} = 1 - 0.4 = 0.6$$

(2)

$$\mathbb{E}[L] = \frac{\rho}{1-\rho} = \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\frac{0.8}{2}}{1-\frac{0.8}{2}} = 0.66$$

Little's law for L and S

Little's law for the average number of jobs in the system and the the average sojourn time reads

$$\mathbb{E}[L] = \lambda\mathbb{E}[S].$$

(3)

$$\mathbb{E}[S] = \frac{\mathbb{E}[L]}{\lambda} = \frac{0.66}{0.8} = 0.833$$

(4)

$$\mathbb{E}[S] = \mathbb{E}[W] + \mathbb{E}[B] = \frac{\rho\mathbb{E}[R]}{1-\rho} + \mathbb{E}[B] = \frac{\rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}}{1-\rho} + \mathbb{E}[B] = \frac{0.40.25}{1-0.4} + 0.5 = 0.66$$

When we compare the answers of (3) and (4) we observe that the expected sojourn time in the answer of (4) is smaller. This is because the service times are deterministic and there is less randomness.

4.3. Road Traffic Analysis

4.3.1. Dijkstra's algorithm

In order to use Dijkstra’s algorithm, we first need to formulate the problem as a shortest route problem. This can be done by replacing the probabilities that are assigned to the edges by the negative of their logarithm. Therefore, the most reliable route of the network in Figure 3.3.1 can be determined by finding the shortest route of the network in Figure 4.3.1.

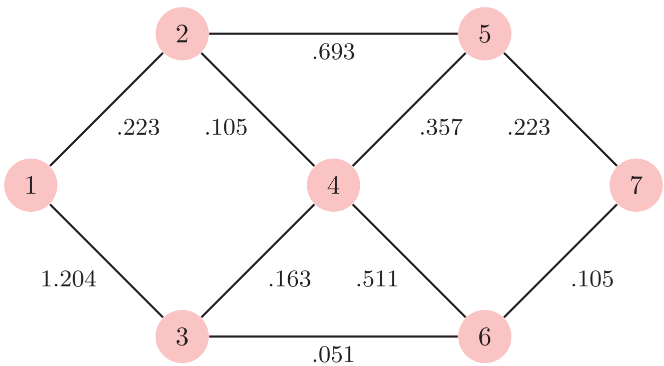


Figure 4.3.1. Most-reliable-route representation as a shortest-route model

Applying Dijkstra’s algorithm gives the route

$$1 \to 2 \to 4 \to 3 \to 6 \to 7,$$

from which it follows that

$$\begin{aligned} -\log p_{17} &= -\log p_{12} - \log p_{24} - \log p_{43} - \log p_{36} - \log p_{67} \\ &= 0.223 + 0.105 + 0.163 + 0.051 + 0.105 \\ &= 0.647. \end{aligned}$$

We conclude

$$p_{17} = e^{-0.647} \approx 0.524.$$

Using this route, there is a probability of 52.4% of not getting stuck in a traffic jam.

4.3.2. Normal Distribution

Since $\mathbb{P}(T \geq t) = 1 - \mathbb{P}(T < t)$, it is sufficient to show

$$\mathbb{P}(T_{13} < 20) > \mathbb{P}(T_{12} < 20).$$

We use that

$$\frac{T_{ij} - \mu_{ij}}{\sigma_{ij}} \sim \mathcal{N}(0, 1).$$

Table 4.0.2 (page 50) then gives

$$\mathbb{P}(T_{13} < 20) = \mathbb{P}\left(\frac{T_{13} - 15}{2} < \frac{5}{2}\right) \approx 0.9938$$

and

$$\mathbb{P}(T_{12} < 20) = \mathbb{P}\left(\frac{T_{12} - 12}{4} < 2\right) \approx 0.9772,$$

from which the conclusion follows.

4.3.3. Road Traffic Network

(1)

$$\begin{aligned} \mathbb{E}[X+Y] &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (x+y) f(x,y) \mathrm{d}y \right) \mathrm{d}x \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} x f(x,y) \mathrm{d}y \right) \mathrm{d}x + \int_{\mathbb{R}} \left(\int_{\mathbb{R}} y f(x,y) \mathrm{d}y \right) \mathrm{d}x \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} x f(x,y) \mathrm{d}y \right) \mathrm{d}x + \int_{\mathbb{R}} \left(\int_{\mathbb{R}} y f(x,y) \mathrm{d}x \right) \mathrm{d}y \\ &= \int_{\mathbb{R}} x \left(\int_{\mathbb{R}} f(x,y) \mathrm{d}y \right) \mathrm{d}x + \int_{\mathbb{R}} y \left(\int_{\mathbb{R}} f(x,y) \mathrm{d}x \right) \mathrm{d}y \\ &= \int_{\mathbb{R}} x f(x) \mathrm{d}x + \int_{\mathbb{R}} y f(y) \mathrm{d}y \\ &= \mathbb{E}[X] + \mathbb{E}[Y], \end{aligned}$$

where used in the third line that we can change the order of integration.

(2) We learned from part (1) that we can simply apply Dijkstra's algorithm to find the route that minimises the sum of the means. This gives the path

$$1 \to 3 \to 4 \to 6 \to 7,$$

which has an expected travel time of 29 minutes.

(3) Using the hint, we find that

$$T_{17} = T_{13} + T_{34} + T_{46} + T_{67} \sim \mathcal{N}(29, 33).$$

Therefore,

$$\mathbb{P}(T_{17} \leq 40) = \mathbb{P}\left(\frac{T_{17} - 29}{\sqrt{33}} \leq \frac{40 - 29}{\sqrt{33}}\right) \approx 0.9722.$$

The alternative route is distributed as

$$\tilde{T}_{17} = T_{13} + T_{35} + T_{57} \sim \mathcal{N}(33, 11).$$

Therefore, the probability of arriving on time is

$$\mathbb{P}(\tilde{T}_{17} \leq 40) = \mathbb{P}\left(\frac{\tilde{T}_{17} - 33}{\sqrt{11}} \leq \frac{40 - 33}{\sqrt{11}}\right) = 0.9826.$$

We observe that the route with the lowest expected travel time is not the route that gives us the highest probability of arriving on time for our job interview.

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