NETWORKS

Communities in Networks

MASTERCLASS

NETWORK

collection of objects that are **interconnected** in ***some*** way

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collection of objects that are **interconnected** in ***some*** way

OBJECTS

- Group of people
- Cities in the Netherlands
- Airports in the Europe

CONNECTION

- Friendship
- Roads
- Flights

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Vertices (Nodes)

CONNECTION

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Edges (Links)

Vertices
(Nodes)

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- Group of people
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Edges
(Links)

A graph is a pair $G = (V, E)$

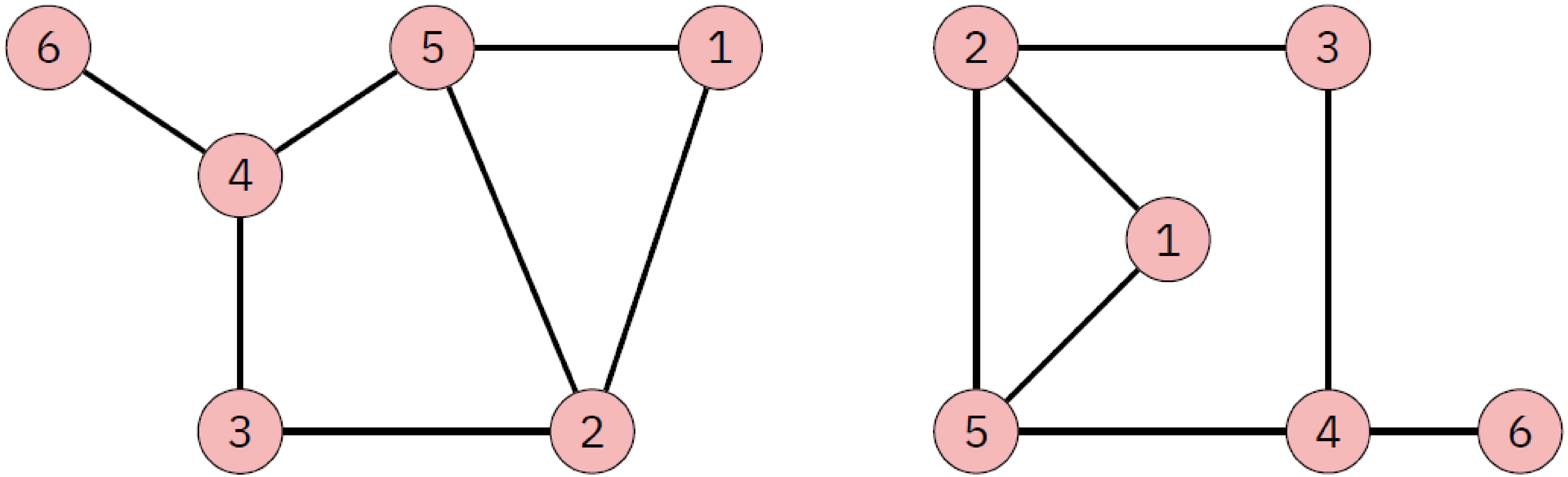
- V: the set of vertices
- E: the set of edges

EXAMPLE 1.3.1. Consider

$$V = \{1, 2, 3, 4, 5, 6\}, \quad E = \{\{1, 2\}, \{1, 5\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{4, 5\}, \{4, 6\}\}.$$

EXAMPLE 1.3.1. Consider

$$V = \{1,2,3,4,5,6\}, \quad E = \{\{1,2\},\{1,5\},\{2,3\},\{2,5\},\{3,4\},\{4,5\},\{4,6\}\}.$$



EXERCISE 1. Show that in a graph $G = (V, E)$ the maximum number of edges is equal to

$$\frac{|V|(|V|-1)}{2}$$

Notation: $|X|$ = the number of elements in a set X

EXERCISE 2. Show that you can construct in total $2^{\binom{|V|}{2}}$ graphs with $|V|$ nodes.

Notation: $\binom{X}{2} = \frac{X(X-1)}{2}$

$\binom{X}{2}$ is also equal to number of pairs in a set X.

$d(x)$: the degree of a node x in a graph $G = (V, E)$

- the number of neighbors of x, that is the number of edges incident to x

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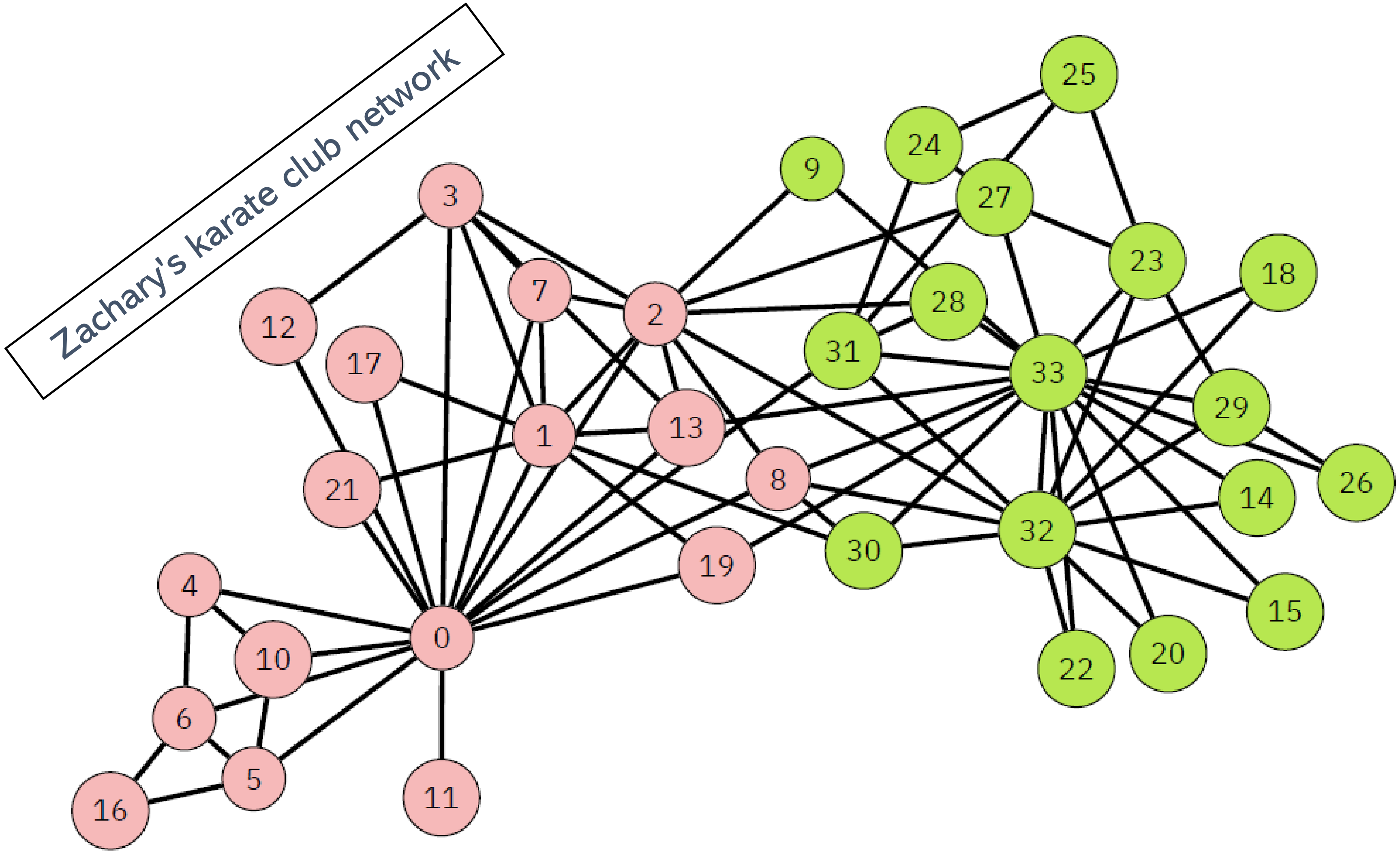

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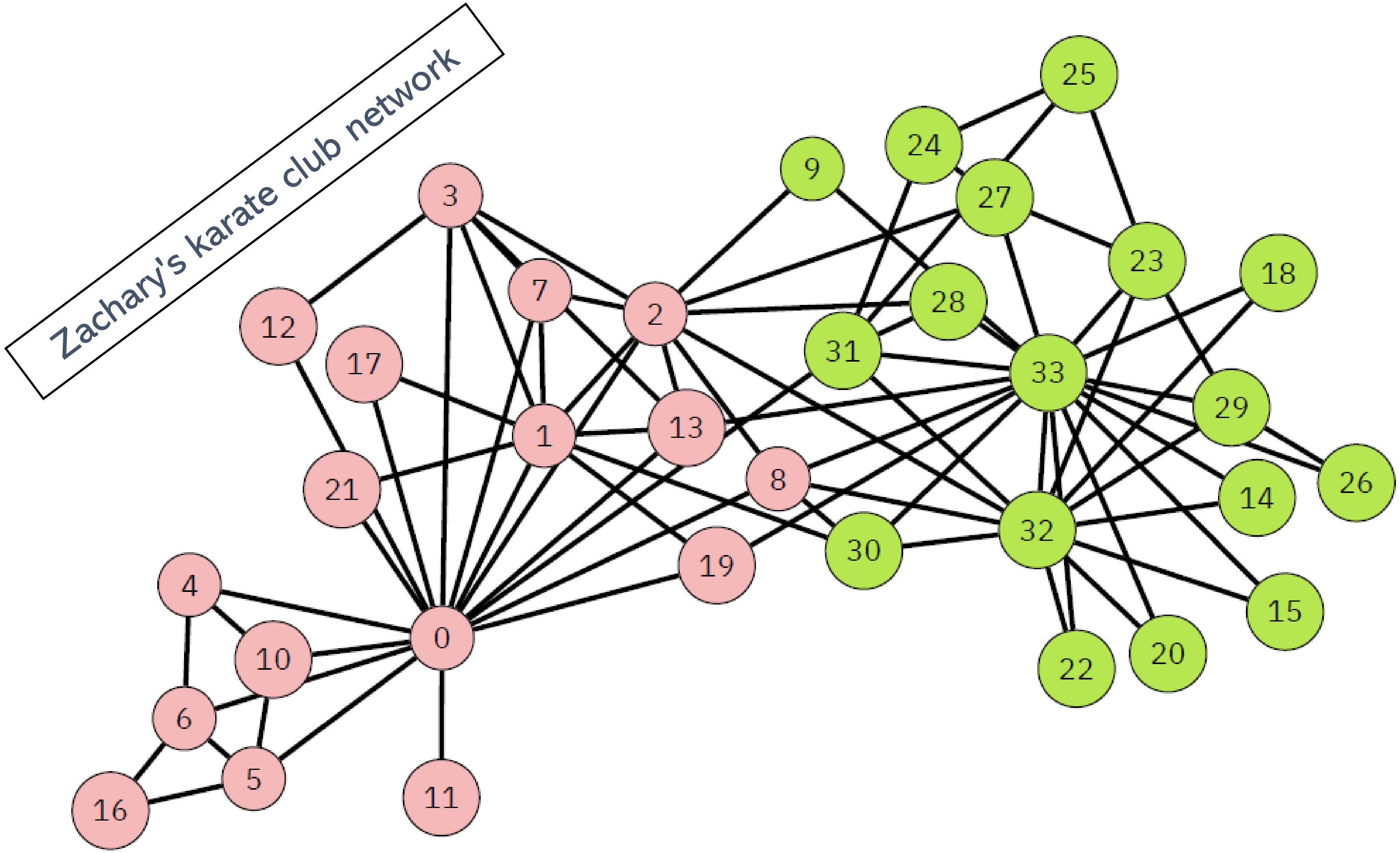

$$d(4) = 3$$

The edges incident to 4 are **{4,0}, {4,6}, and {4,10}**. In other words, neighbors of 4 are **0,6, and 10**.

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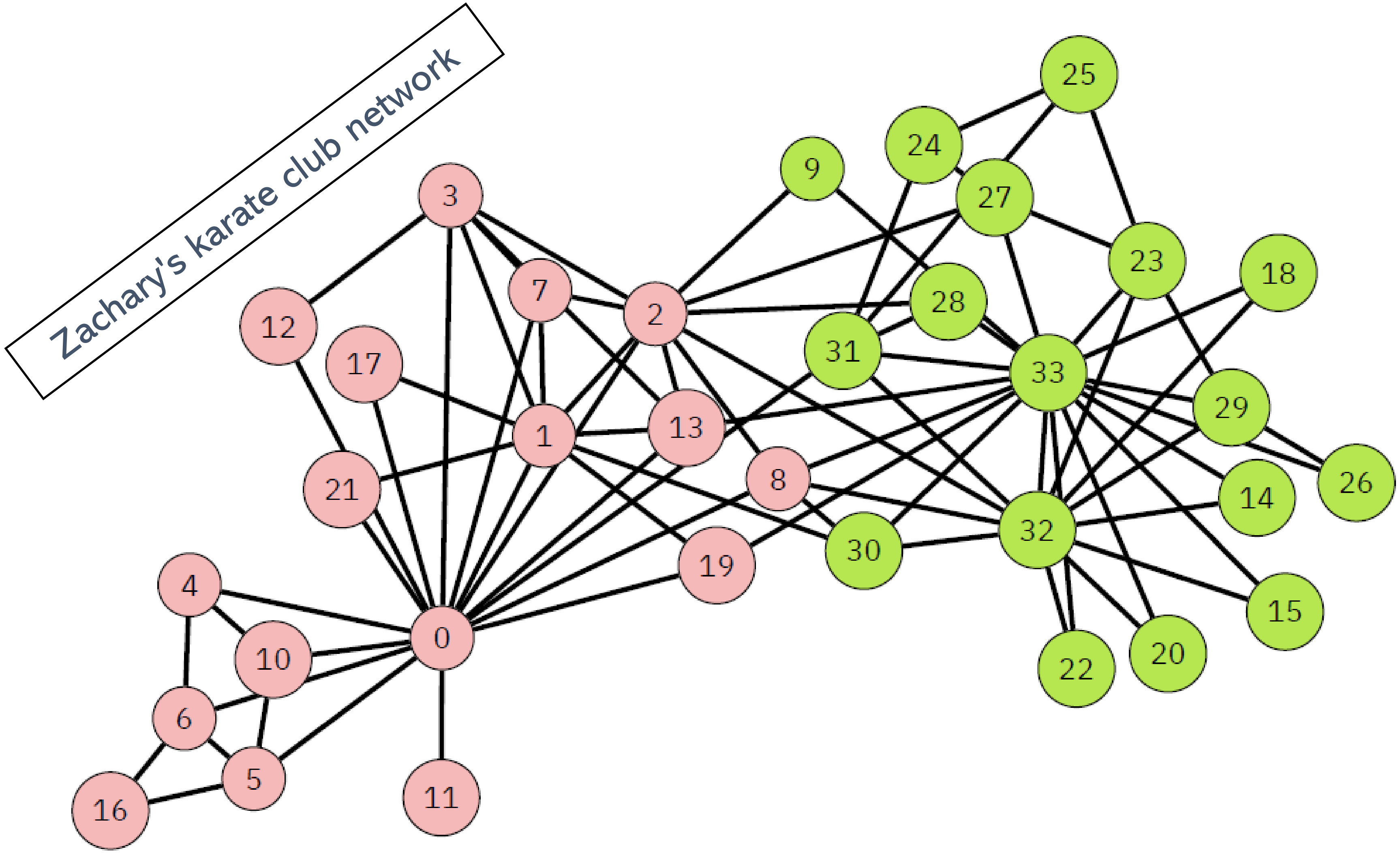

$d(10) = ?$
$d(11) = ?$
$d(22) = ?$
$d(0) = ?$

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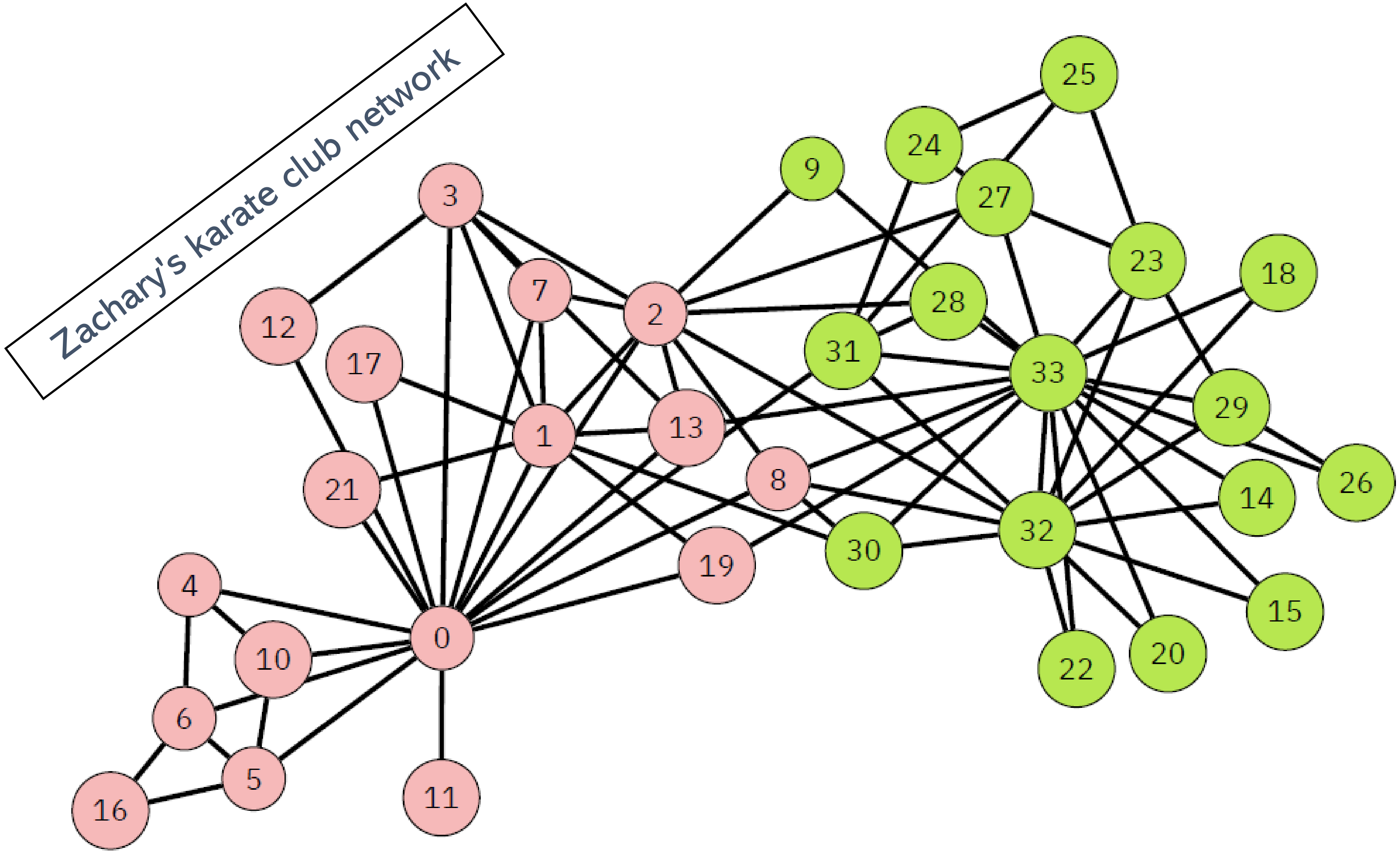
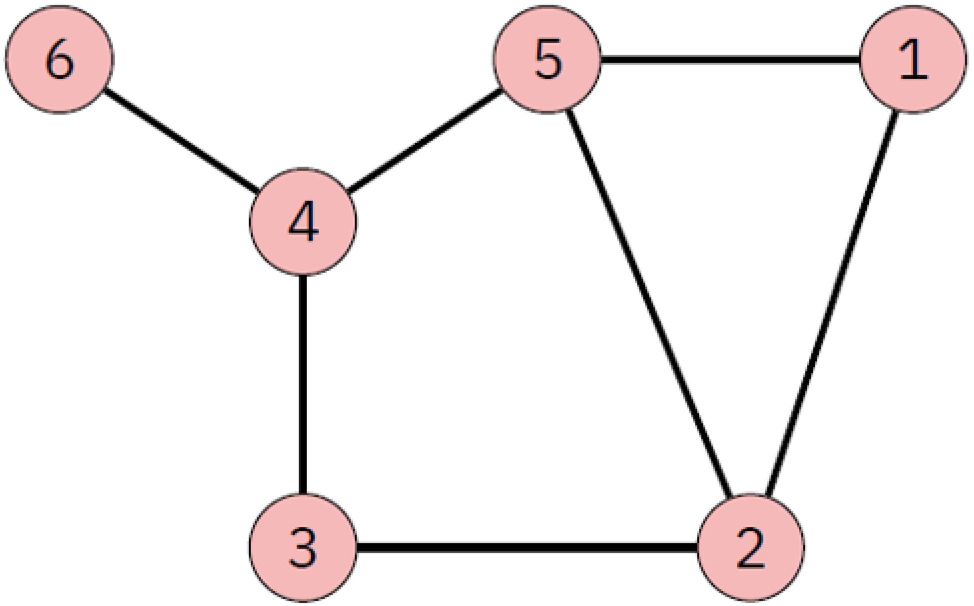


$d(10) = 3$
$d(11) = 1$
$d(22) = 2$
$d(0) = 16$

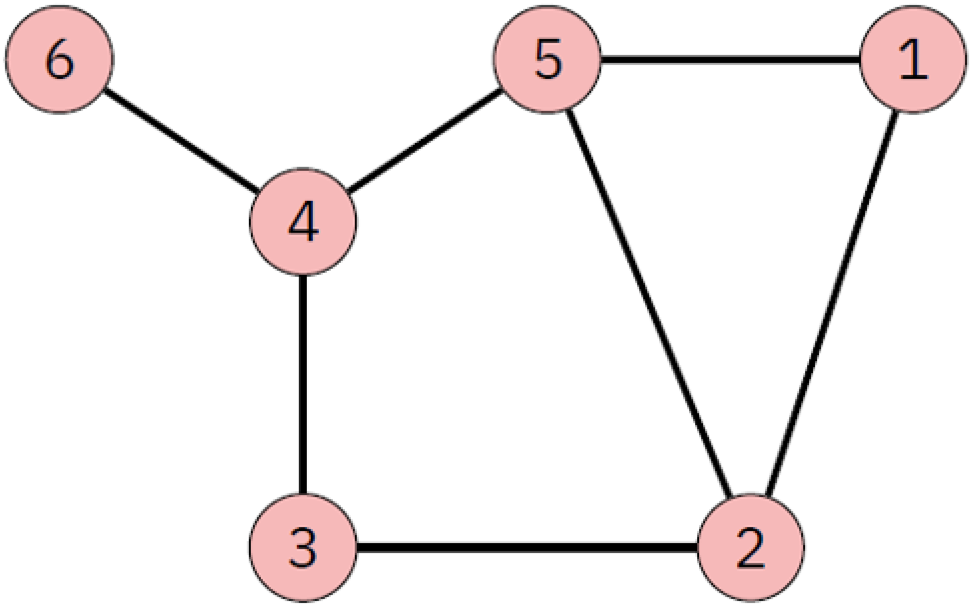


sum of degrees

$$d(1) + d(2) + d(3) + d(4) + d(5) + d(6)$$

$$V = \{1, 2, 3, 4, 5, 6\}$$

$$E = \{\{1, 2\}, \{1, 5\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{4, 5\}, \{4, 6\}\}$$



sum of degrees

$$d(1) + d(2) + d(3) + d(4) + d(5) + d(6)$$

$$2 + 3 + 2 + 3 + 3 + 1$$

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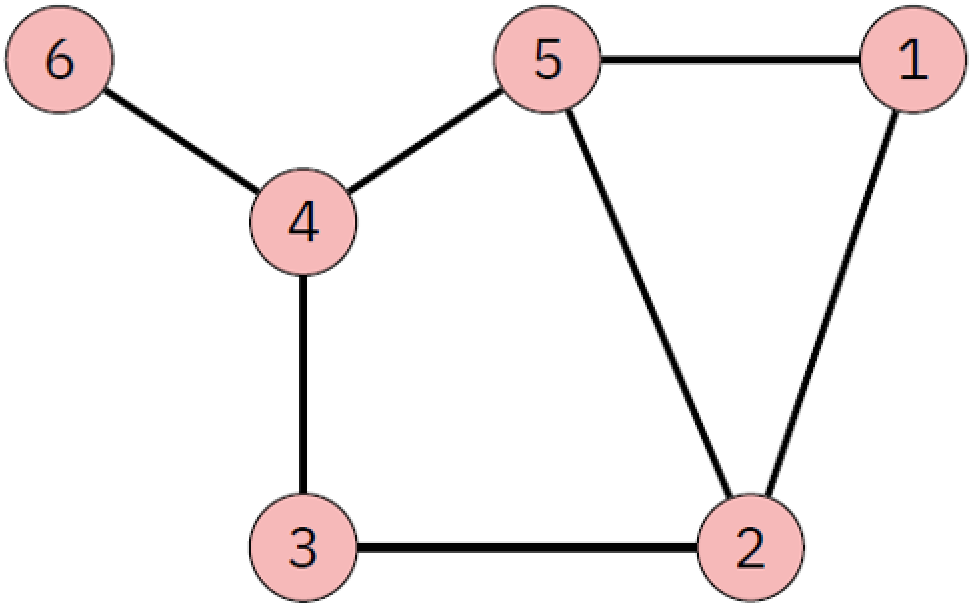

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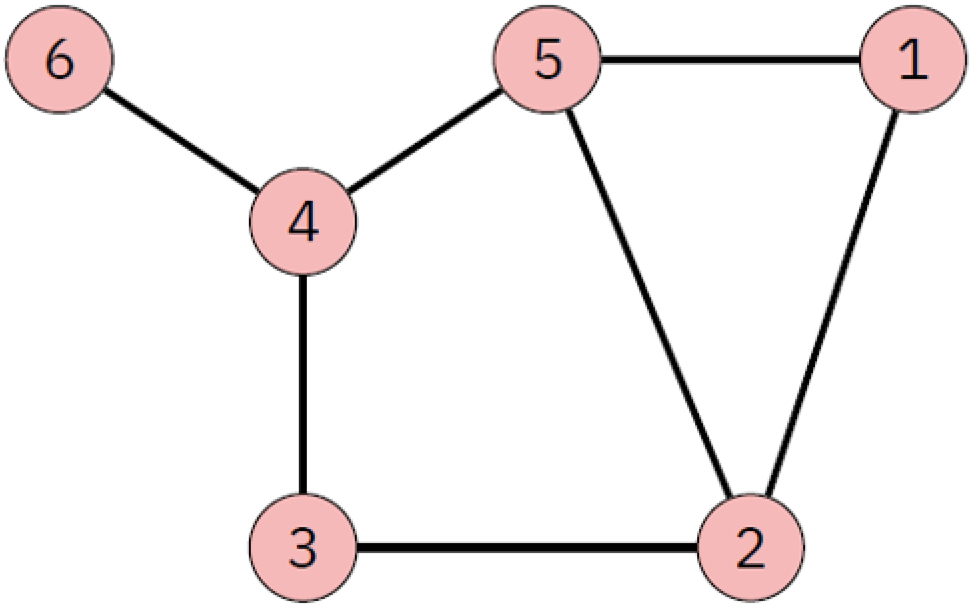
sum of degrees

$$d(1) + d(2) + d(3) + d(4) + d(5) + d(6)$$

$$2 + 3 + 2 + 3 + 3 + 1$$

$$\boxed{\sum_{i=1}^{6} d(i) = 14}$$



$V = \{1, 2, 3, 4, 5, 6\}$

$E = \{\{1,2\}, \{1,5\}, \{2,3\}, \{2,5\}, \{3,4\}, \{4,5\}, \{4,6\}\}$

sum of degrees

$d(1) + d(2) + d(3) + d(4) + d(5) + d(6)$

$2 + 3 + 2 + 3 + 3 + 1$

$$\boxed{\sum_{i=1}^{6} d(i) = 14}$$

$$\boxed{|E| = \ell(G) = 7}$$

Handshaking Lemma

For any graph $G = (V, E)$

$$\sum_{i \in V} d(i) = 2|E| = 2\ell(G)$$

Handshaking Lemma

For any graph $G = (V, E)$

$$\sum_{i \in V} d(i) = 2|E| = 2\ell(G)$$

Proof: Each edge $\{a, b\}$ of the graph $G = (V, E)$ contributes the degree of a and b by one. In other words, each edge $\{a, b\}$ is counted on the left-hand side summation twice:

one in $d(a)$ and one in $d(b)$

Exercise

If possible, draw a graph with degrees given. If it is not possible, explain!

a) 1,2,3,3,3,3

b) 0,1,2,3,4,5,6,7

c) 1,2,2,3,3,3,4

A clique in a graph $G = (V, E)$

- A group of nodes where **each node has an edge to each other node** in the group.

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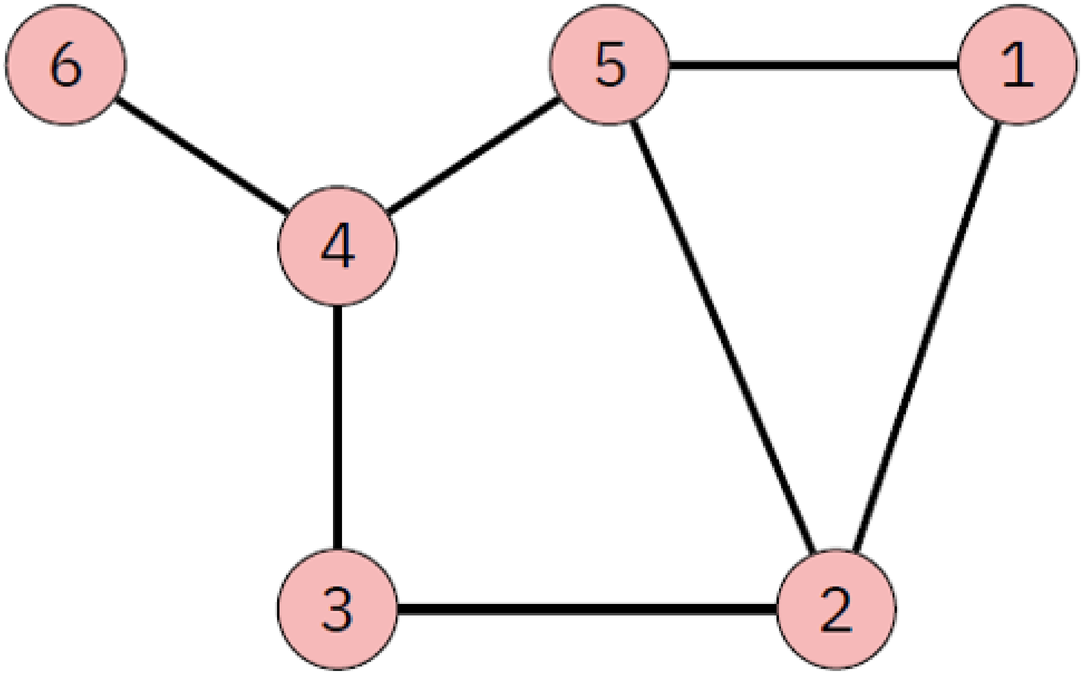

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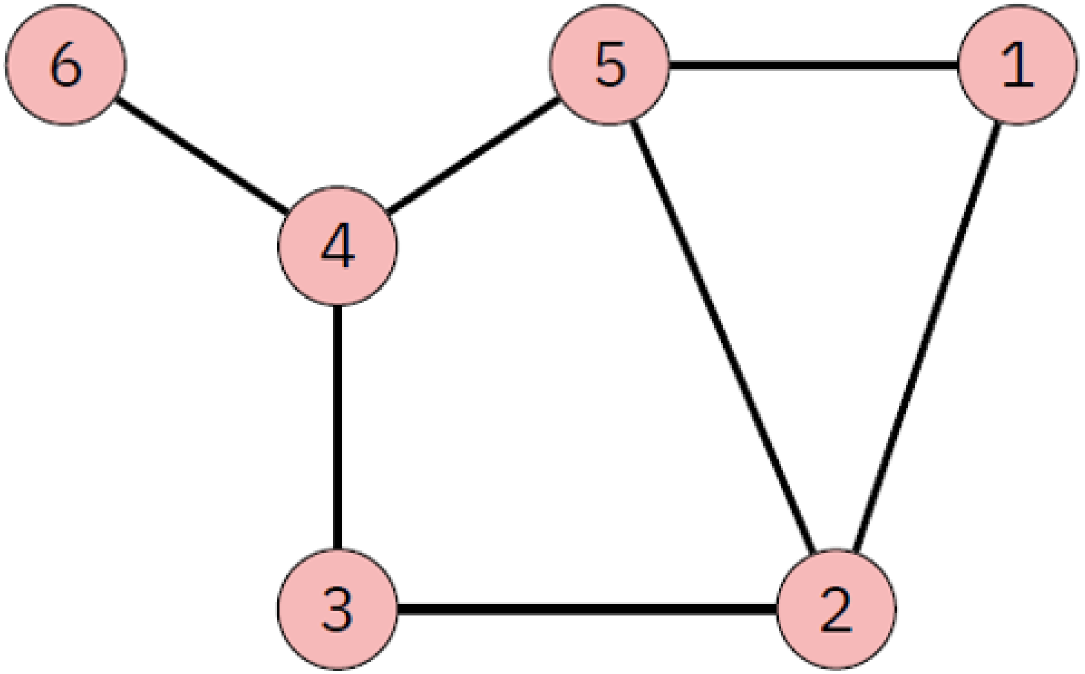

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Any single edge is also a clique, e.g. {2,3}, {4,6}

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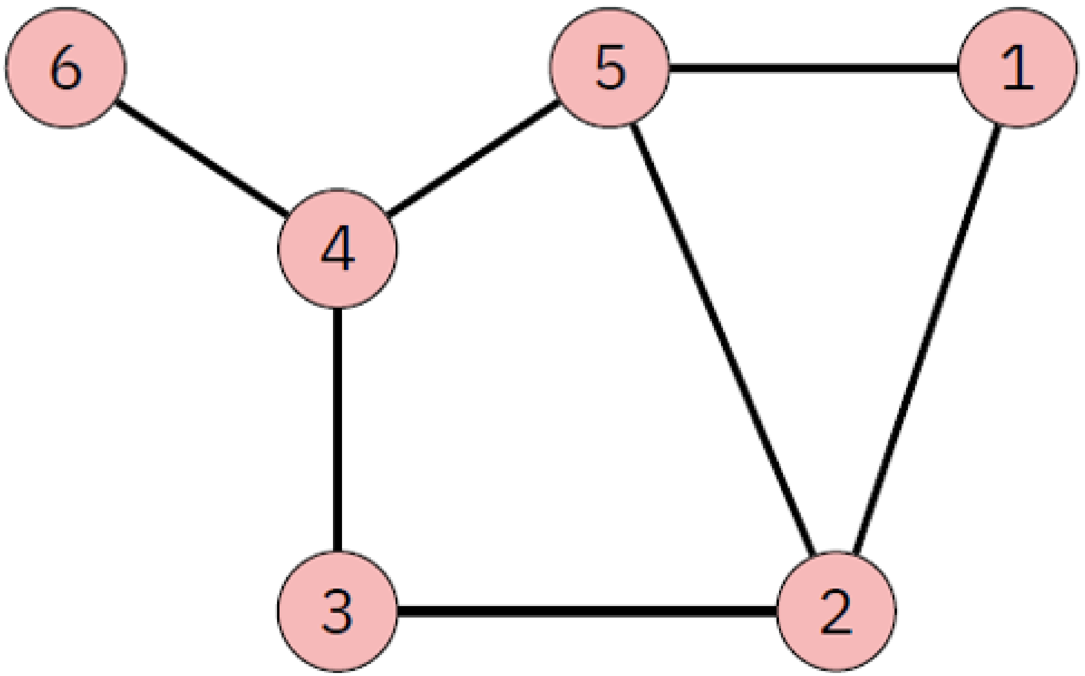
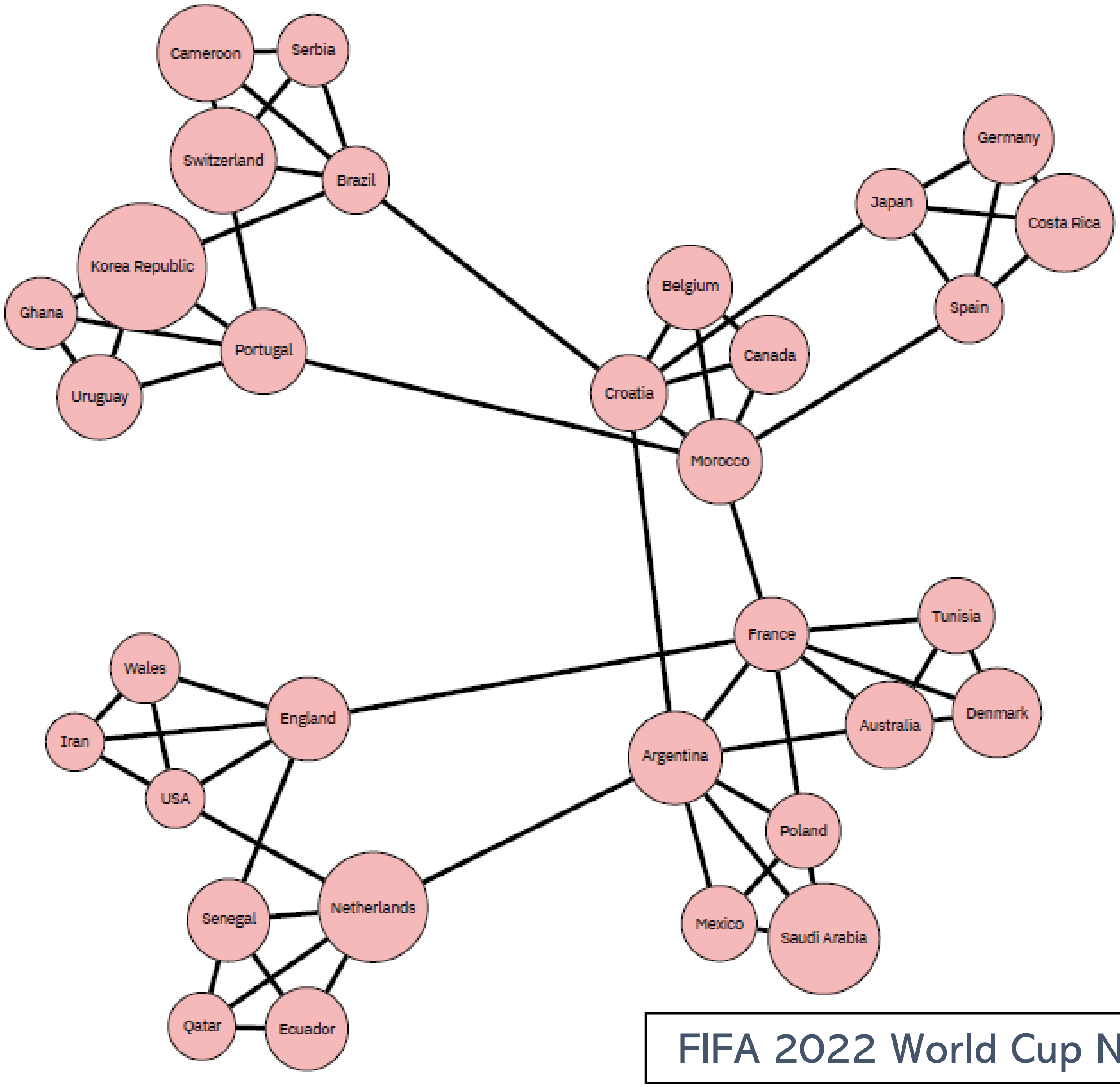


$\{1,2,5\}$ forms a clique because $\{1,2\}$, $\{1,5\}$, and $\{2,5\}$ are the edges of the graph

Any single edge is also a clique, e.g. $\{2,3\}$, $\{4,6\}$

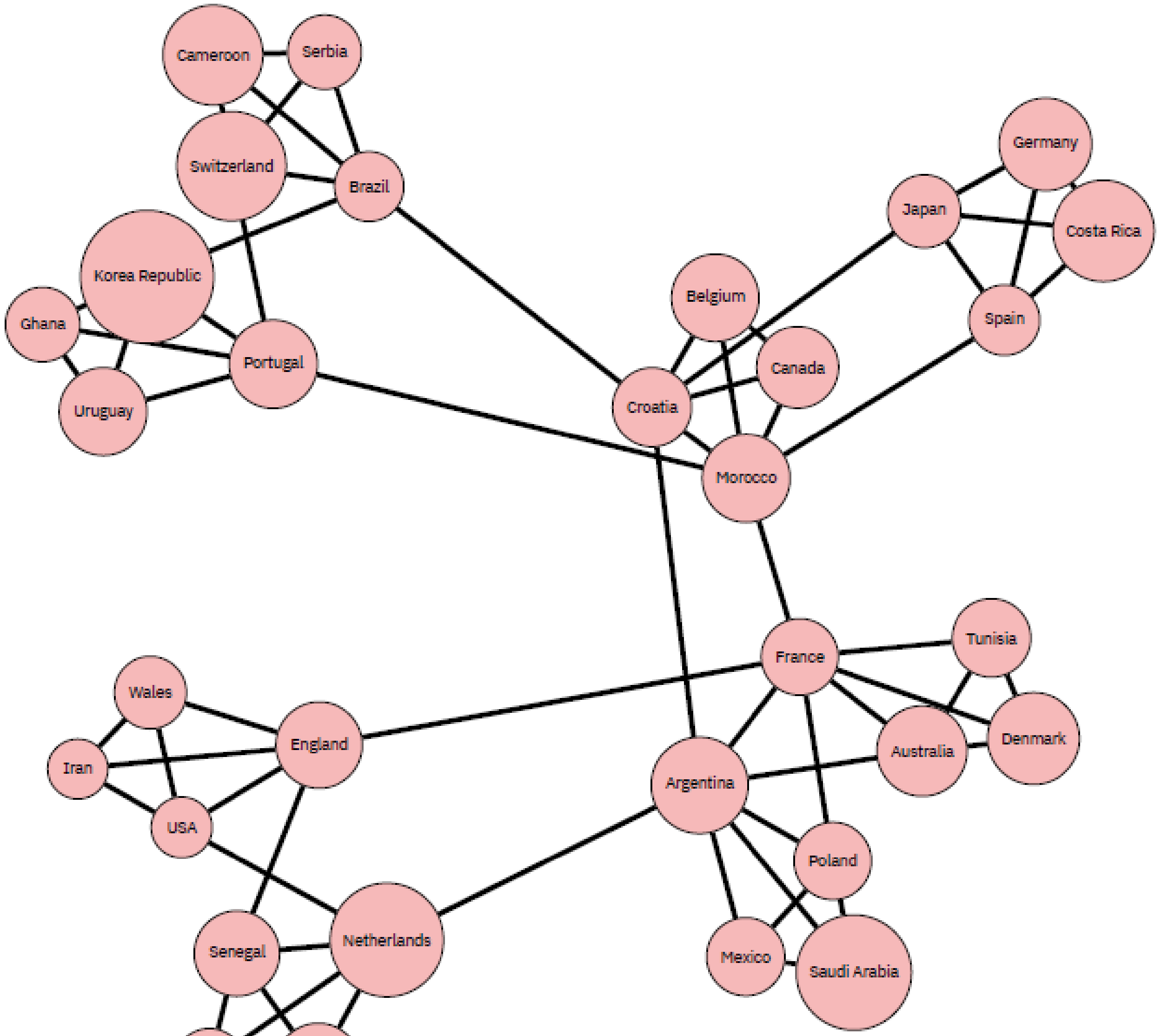
$\{2,3,4\}$ is not a clique because $\{2,4\}$ is not an edge

EXERCISE 5. What is the size of the largest clique of the FIFA2022 network



FIFA 2022 World Cup Network

EXERCISE 5. What is the size of the largest clique of the FIFA2022 network

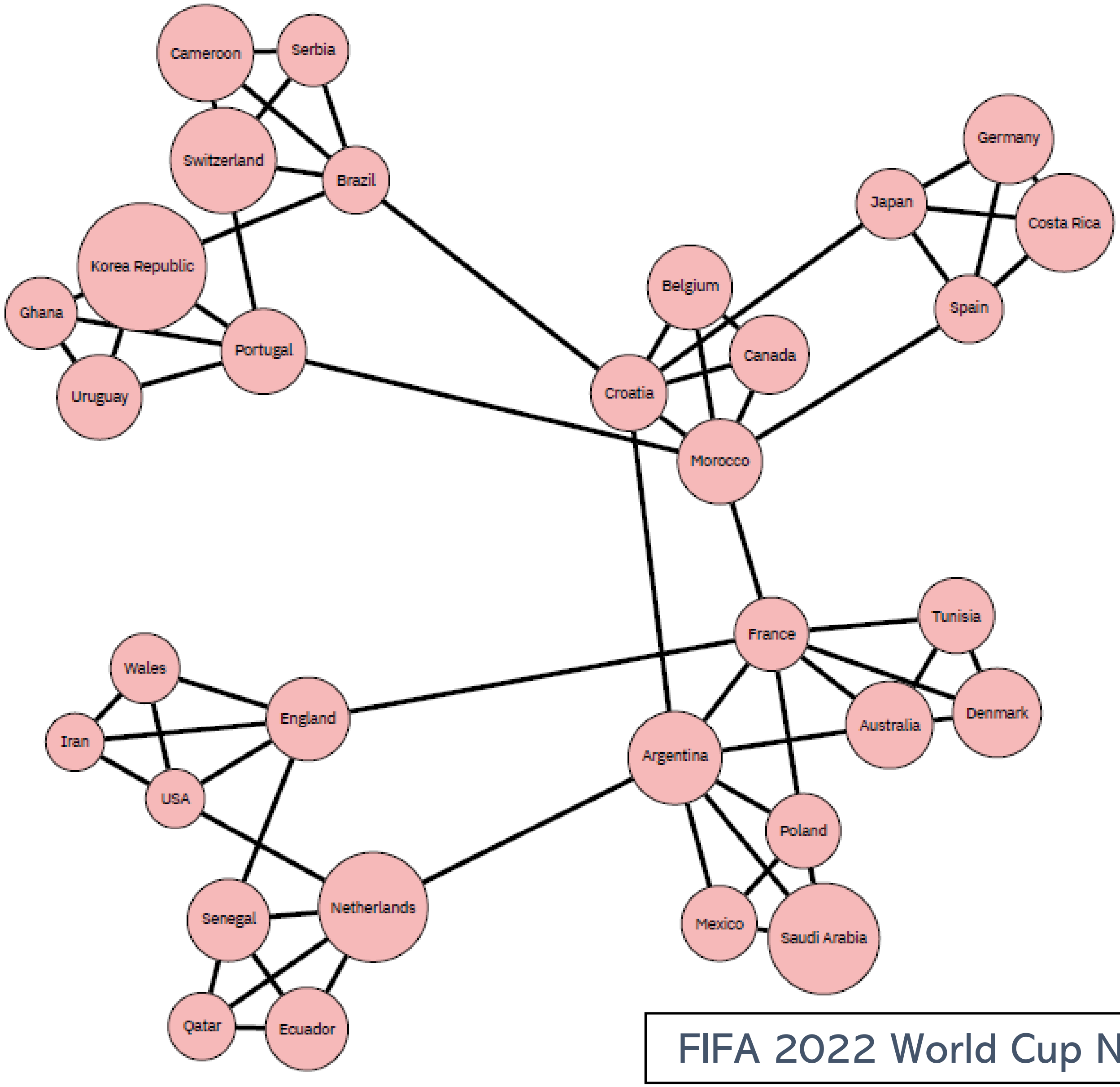


FIFA 2022 World Cup Network

Each group forms a clique of size 4 as each team plays with each other team within the group, for instance

{Netherlands, Senegal, Qatar, Ecuador}

EXERCISE 5. What is the size of the largest clique of the FIFA2022 network



FIFA 2022 World Cup Network

Each group forms a clique of size 4 as each team plays with each other team within the group, for instance

{Netherlands, Senegal, Qatar, Ecuador}

There is no clique of size 5!

So, the size of the largest clique is 4.

A **path** between the nodes a and b

- a **sequence of edges** which joins a sequence of nodes **from $\boldsymbol{a}$ to $\boldsymbol{b}$**

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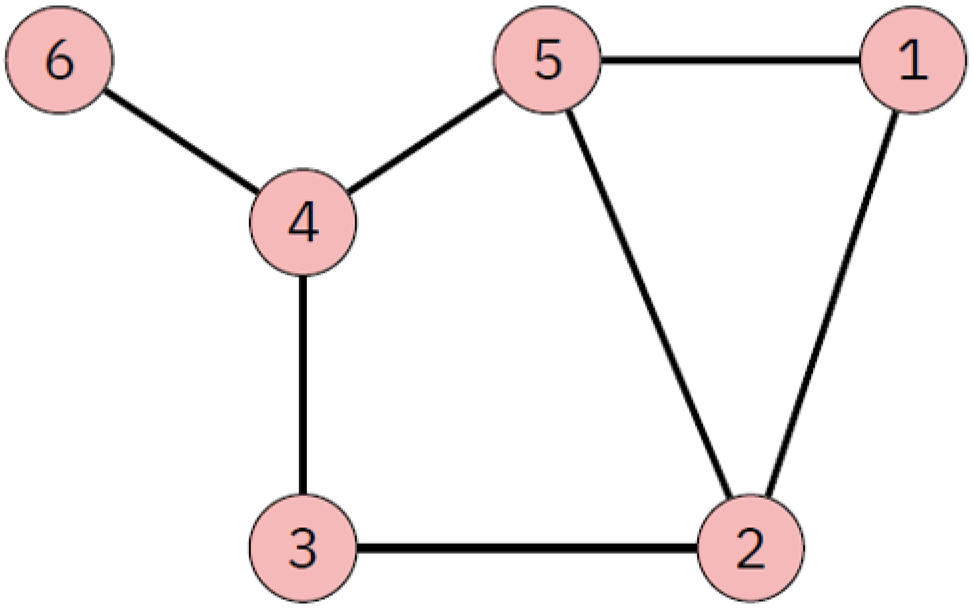

$6 \to 4 \to 3 \to 2$ forms a path from 6 to 2.

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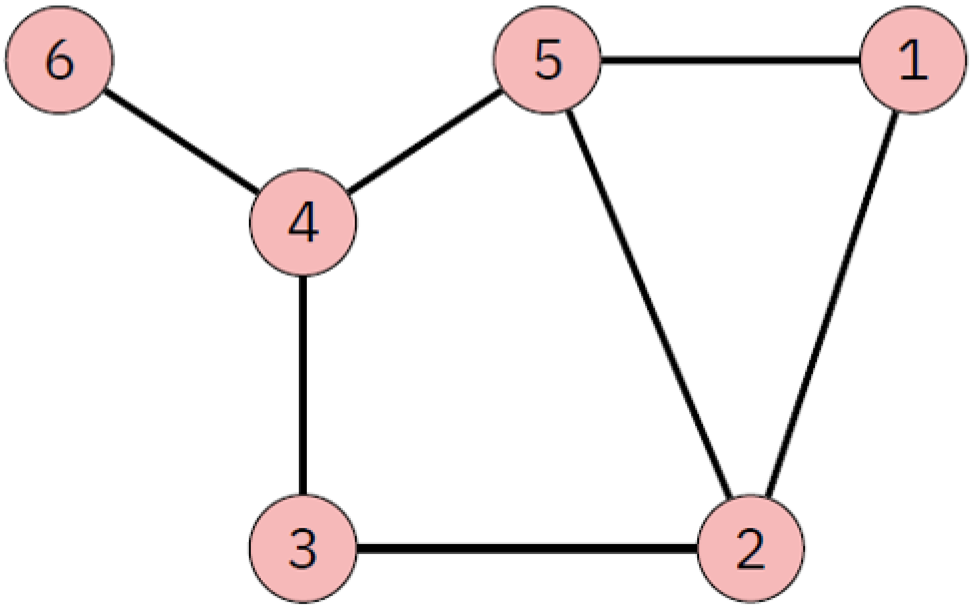

$6 \to 4 \to 3 \to 2$ forms a path from 6 to 2.

$6 \to 4 \to 3 \to 1$ is not a path since $\{3, 1\}$ is not an edge

A **shortest path** between the nodes a and b

A **path** using the **least** amount of edges

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Distance: the **number of edges** on the shortest path, i.e. **the length** of the shortest path

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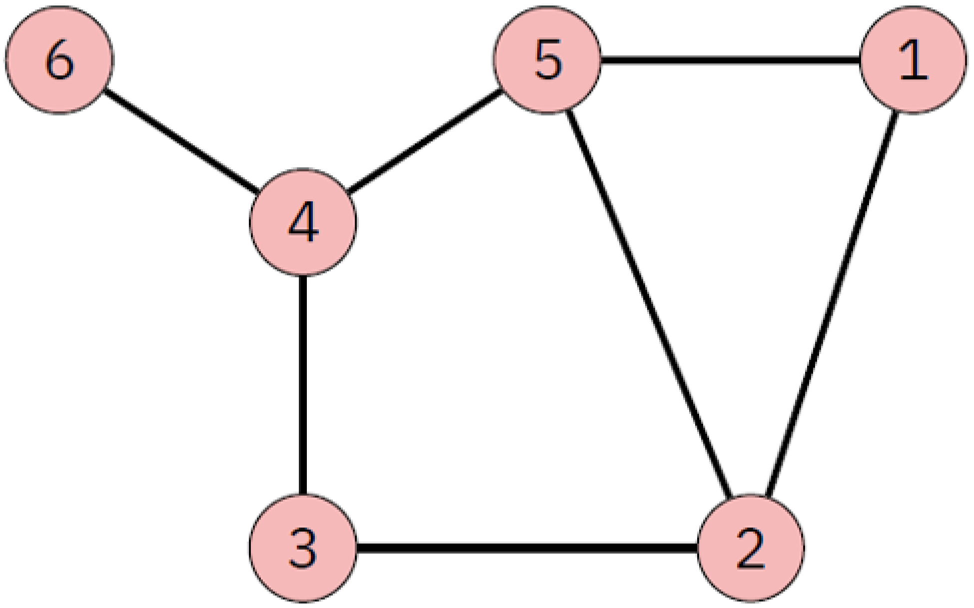

Distance: the **number of edges** on the shortest path, i.e. **the length** of the shortest path

$$Dist(6,1) = 3$$

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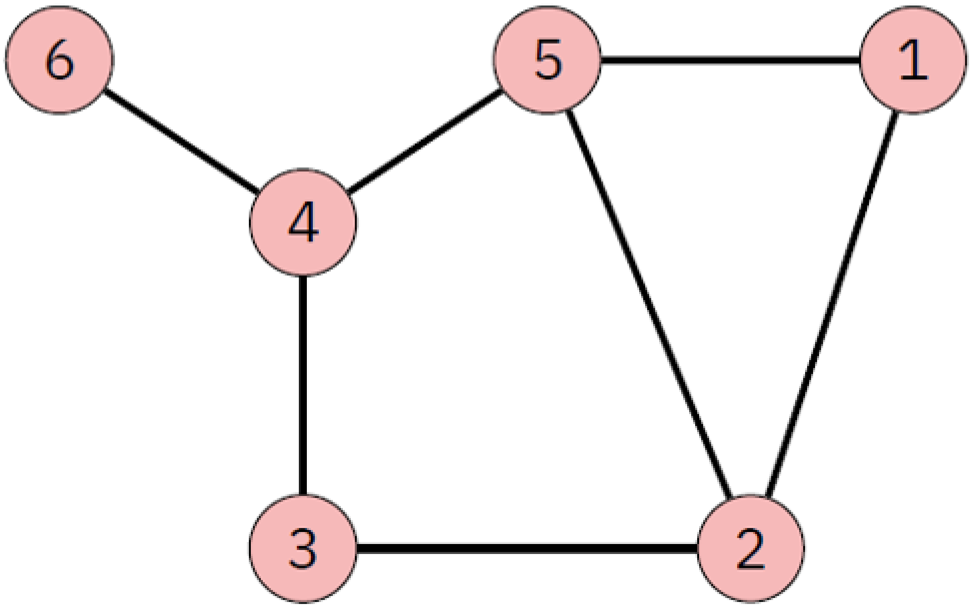

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$$Dist(6,1) = 3$$

$6 \to 4 \to 5 \to 1$ is a shortest path between 6 and 1

$6 \to 4 \to 3 \to 2 \to 1$ and $6 \to 4 \to 3 \to 2 \to 5 \to 1$ are also paths between 6 and 1, but they are longer.

A **shortest path** between the nodes a and b

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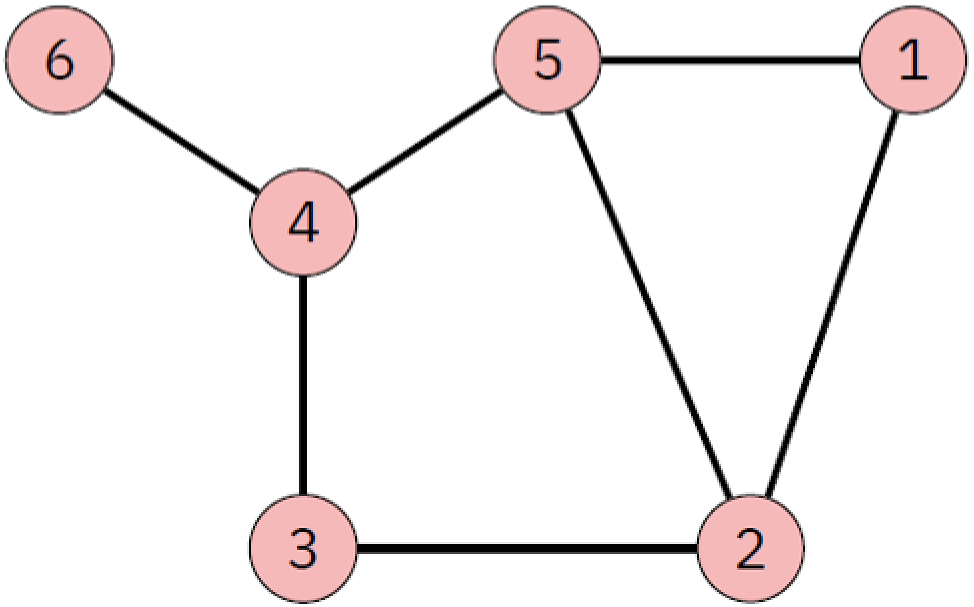

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$$Dist(1,2) = ?$$

$$Dist(1,4) = ?$$

$$Dist(2,6) = ?$$

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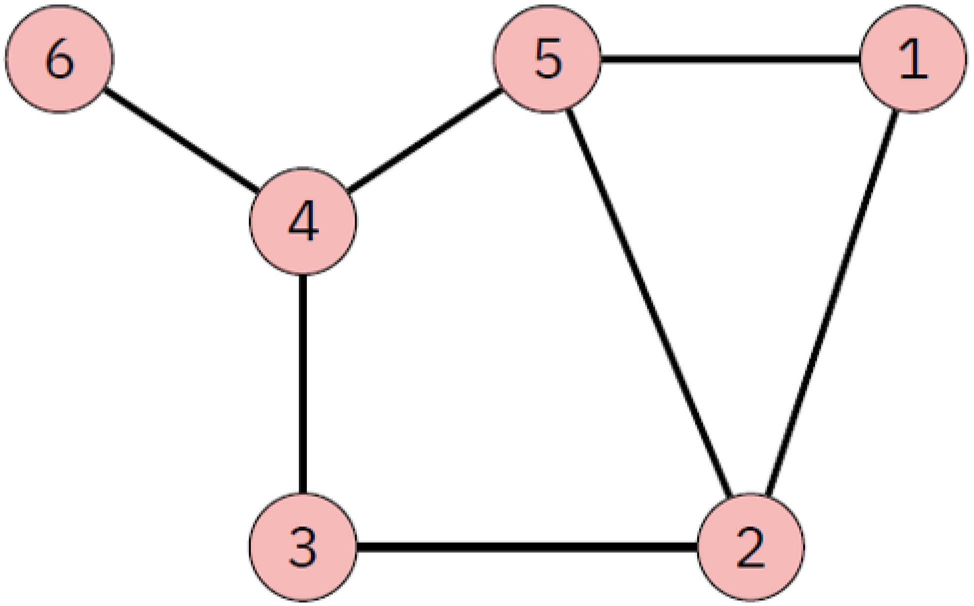

Distance: the **number of edges** on the shortest path, i.e. **the length** of the shortest path

$$Dist(1,2) = 1$$

$$Dist(1,4) = 2$$

$$Dist(2,6) = 3$$

Connected vs Disconnected

- G is **connected** if there exists a path between any pair of vertices
- G is **disconnected** if there exists a pair of vertices with no path between them

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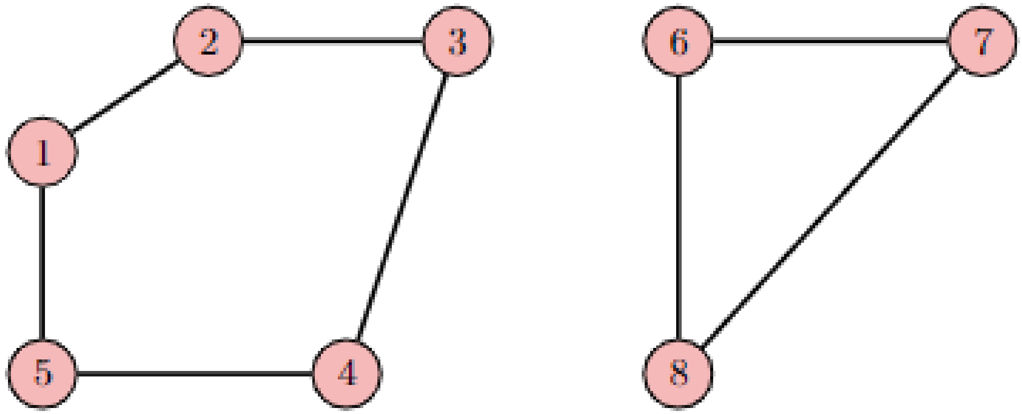

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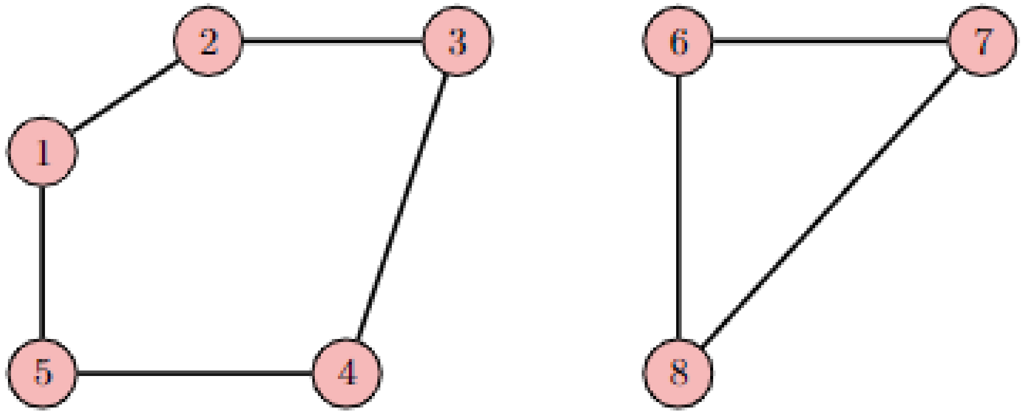

$$Dist(1,4) = 2$$

$$Dist(6,7) = 1$$

$$Dist(2,6) = \infty$$

Further Definitions

- Let S be a part of the graph (possibly whole graph)
- For any vertex i in the graph, the eccentricity of i is defined as

$$\boxed{\mathsf{Ecc}(i, S) = \max_{j \in S} \mathsf{Dist}(i, j)}$$

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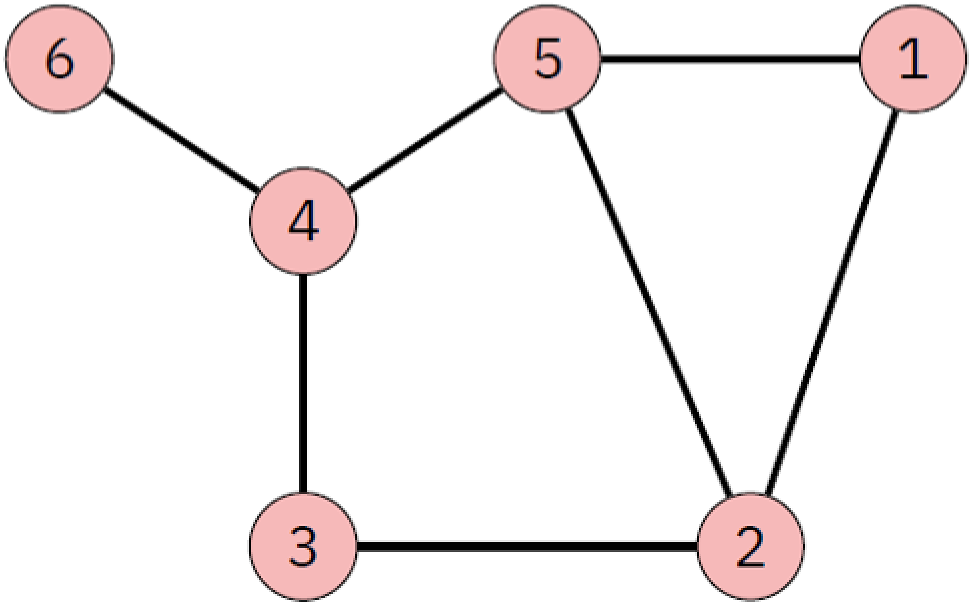

$$Ecc(1, \{2,3,4\}) = ?$$

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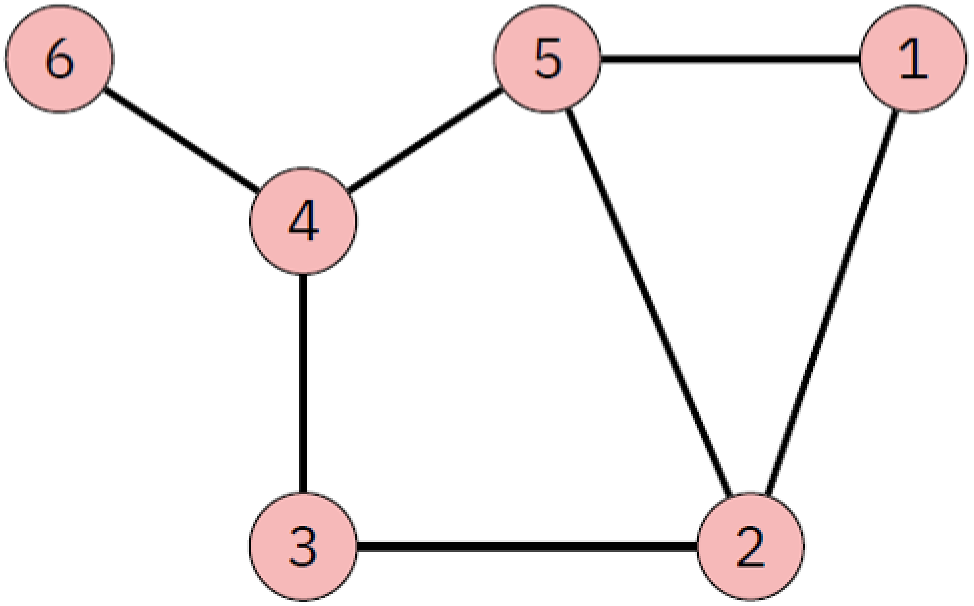

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$Dist(1,2)$

$Dist(1,3)$

$Dist(1,4)$

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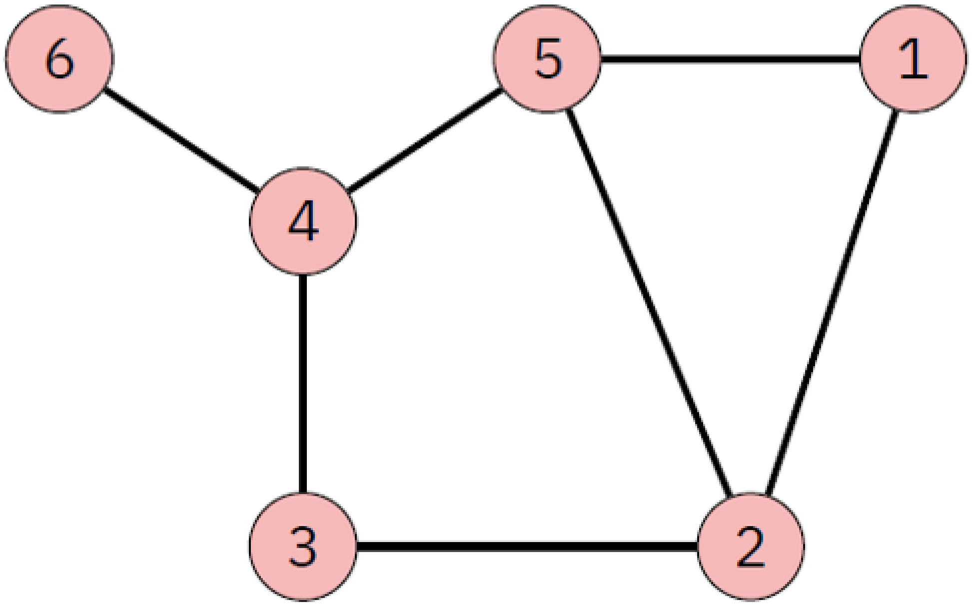

$$Ecc(1, \{2,3,4\}) = ?$$

$$Dist(1,2) = 1$$

$$Dist(1,3) = 2$$

$$Dist(1,4) = 2$$

$$Ecc(1, \{2,3,4\}) = \max\{1,2,2\} = 2$$

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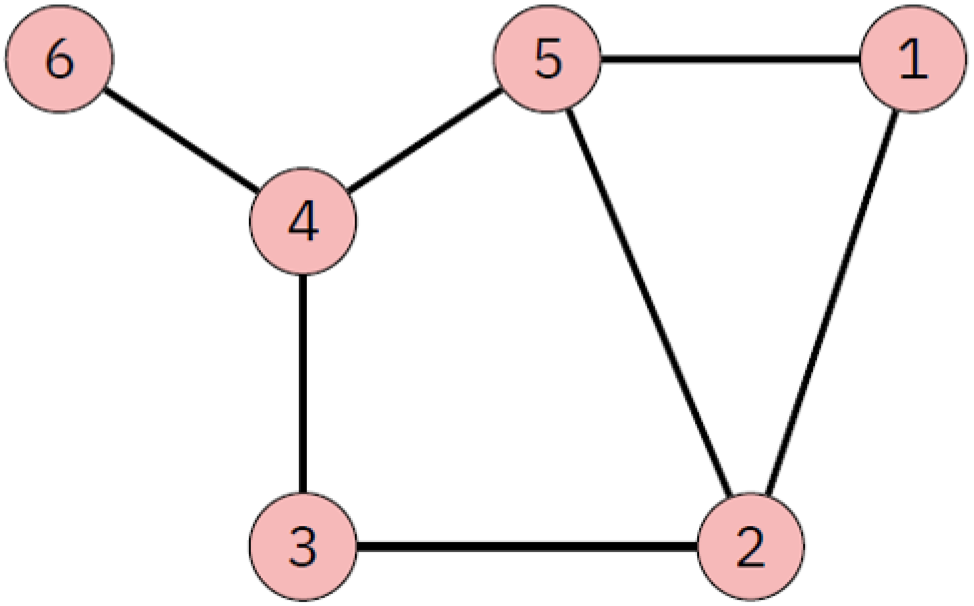

$$Ecc(1, \{2,3,4\}) = 2$$

$$Ecc(1, G) = ?$$

$$Ecc(4, \{3,4,5,6\}) = ?$$

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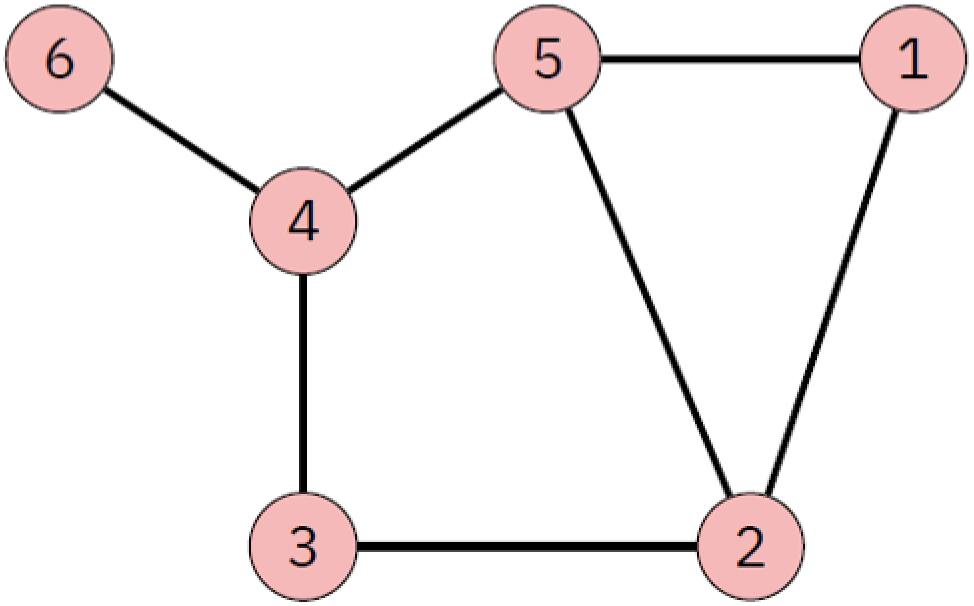

$$Ecc(1, \{2,3,4\}) = 2$$

$$Ecc(1, G) = 3$$

$$Ecc(4, \{3,4,5,6\}) = 1$$

Further Definitions

- Let S be a part of the graph (possibly whole graph)
- The diameter of S is defined as

$$\boxed{\text{Diameter}(S) = \max_{i \in S} \text{Ecc}(i, S)}$$

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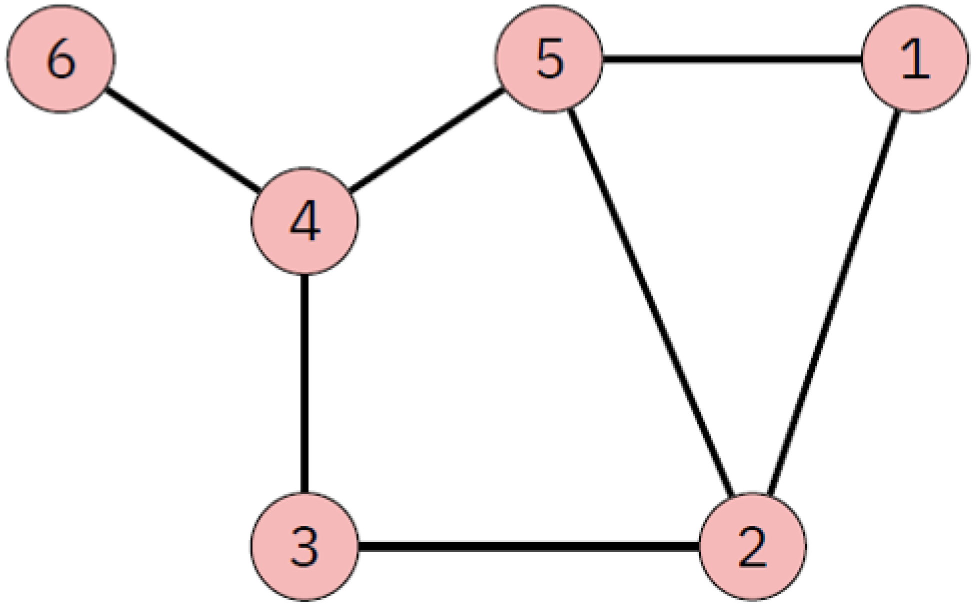

$$Diameter(\{2,3,4\}) = ?$$

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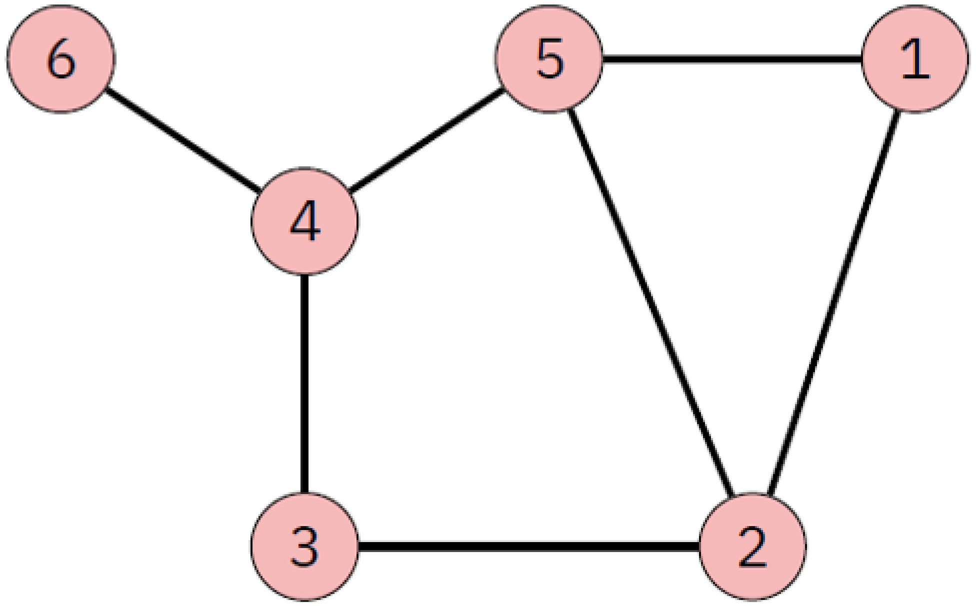

$$Diameter(\{2,3,4\}) = ?$$

$$Ecc(2, \{2,3,4\})$$

$$Ecc(3, \{2,3,4\})$$

$$Ecc(4, \{2,3,4\})$$

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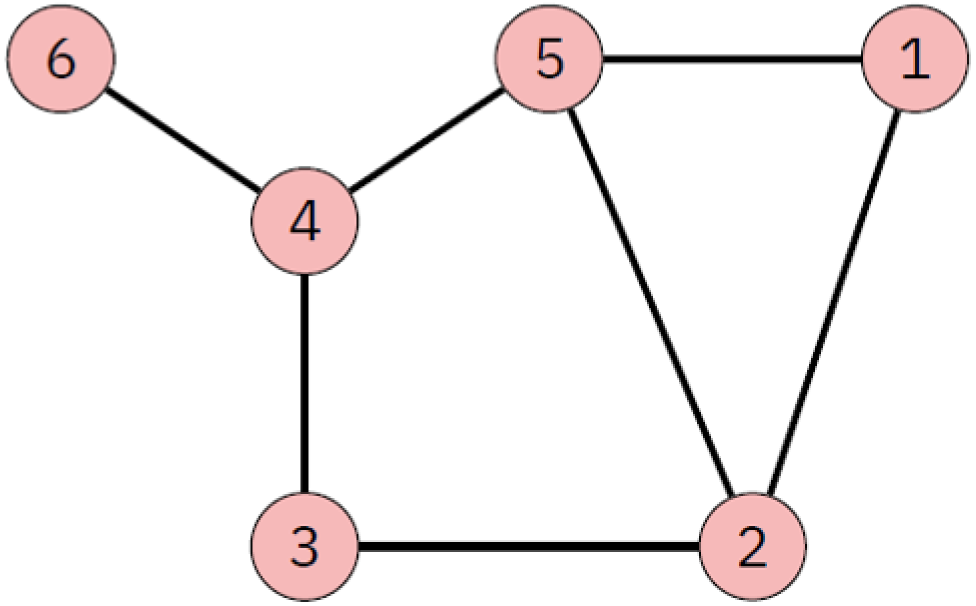

$$Diameter(\{2,3,4\}) = ?$$

$$Ecc(2, \{2,3,4\}) = \max\{Dist(2,2), Dist(2,3), Dist(2,4)\}$$

$$Ecc(3, \{2,3,4\}) = \max\{Dist(3,2), Dist(3,3), Dist(3,4)\}$$

$$Ecc(4, \{2,3,4\}) = \max\{Dist(4,2), Dist(4,3), Dist(4,3)\}$$

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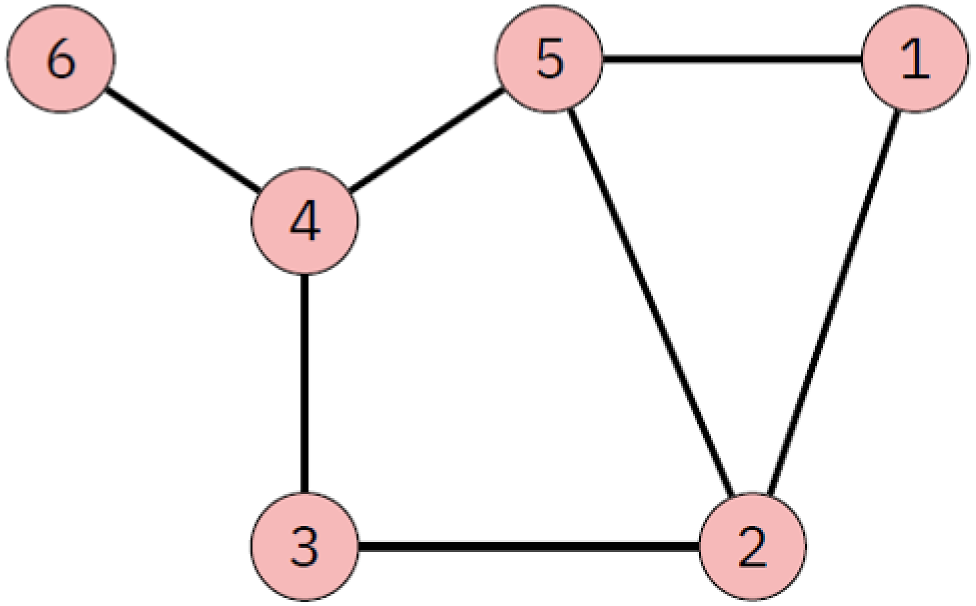

$$Diameter(\{2,3,4\}) = ?$$

$$Ecc(2, \{2,3,4\}) = \max\{Dist(2,2), Dist(2,3), Dist(2,4)\} = 2$$

$$Ecc(3, \{2,3,4\}) = \max\{Dist(3,2), Dist(3,3), Dist(3,4)\} = 1$$

$$Ecc(4, \{2,3,4\}) = \max\{Dist(4,2), Dist(4,3), Dist(4,3)\} = 2$$

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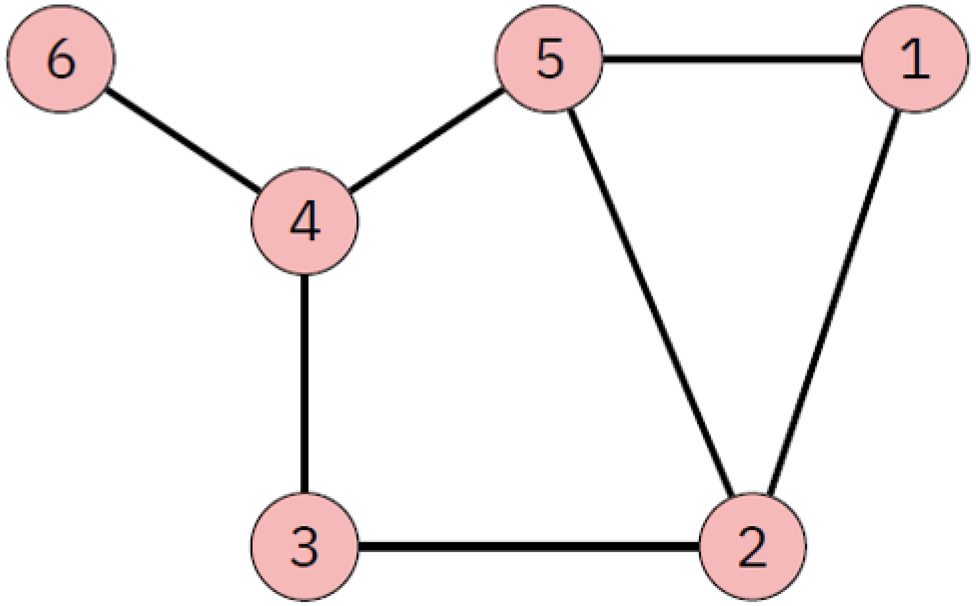

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$$Diameter(\{2,3,4\}) = \max\{2,1,2\} = 2$$

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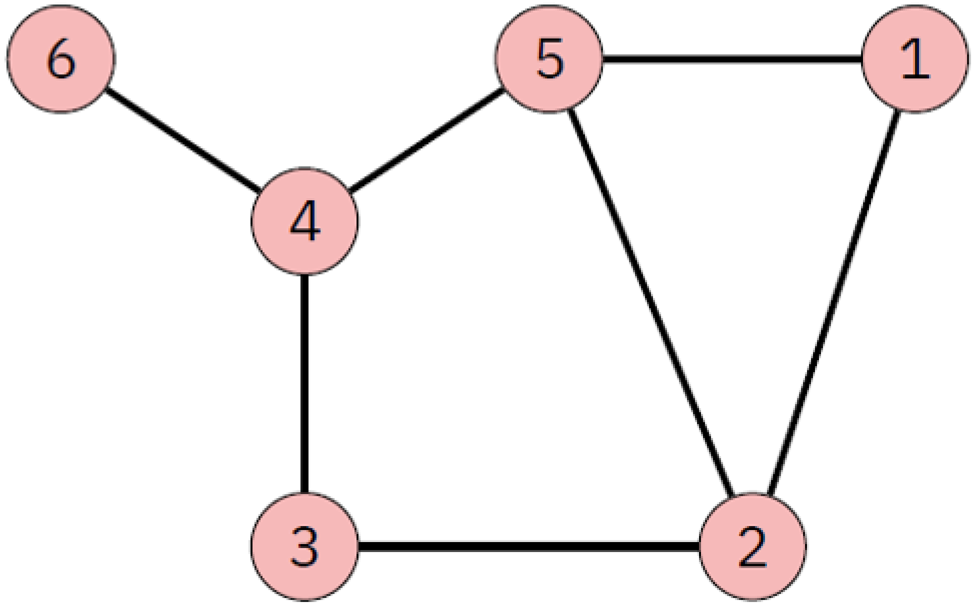
- The diameter of S is defined as

$$\boxed{\mathrm{Diameter}(S) = \max_{i \in S} \mathrm{Ecc}(i, S)}$$



$$Diameter(\{2,3,4\}) = 2$$

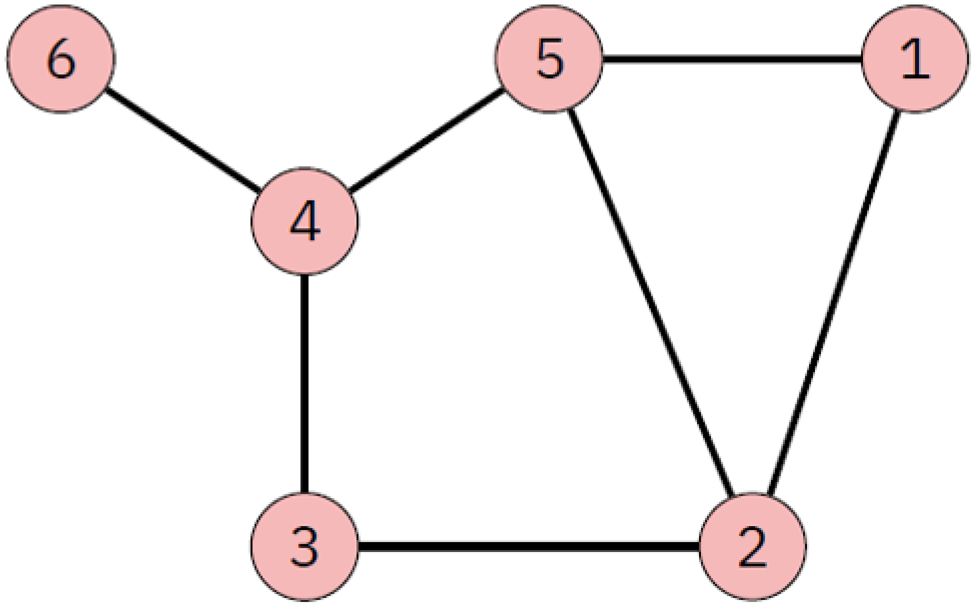
$$Diameter(\{2,3,6\}) = ?$$

$$Diameter(G) = ?$$

Further Definitions

- Let S be a part of the graph (possibly whole graph)
- The diagram of S is defined as

$$\text{Diameter}(S) = \max_{i \in S} \text{Ecc}(i, S)$$



$$Diameter(\{2,3,4\}) = 2$$

$$Diameter(\{2,3,6\}) = 3$$

$$Diameter(G) = 3$$

Further Definitions

- Let S be a part of the graph (possibly whole graph)
- The radius of S is defined as

$$\boxed{\text{Radius}(S) = \min_{i \in S} \text{Ecc}(i, S)}$$

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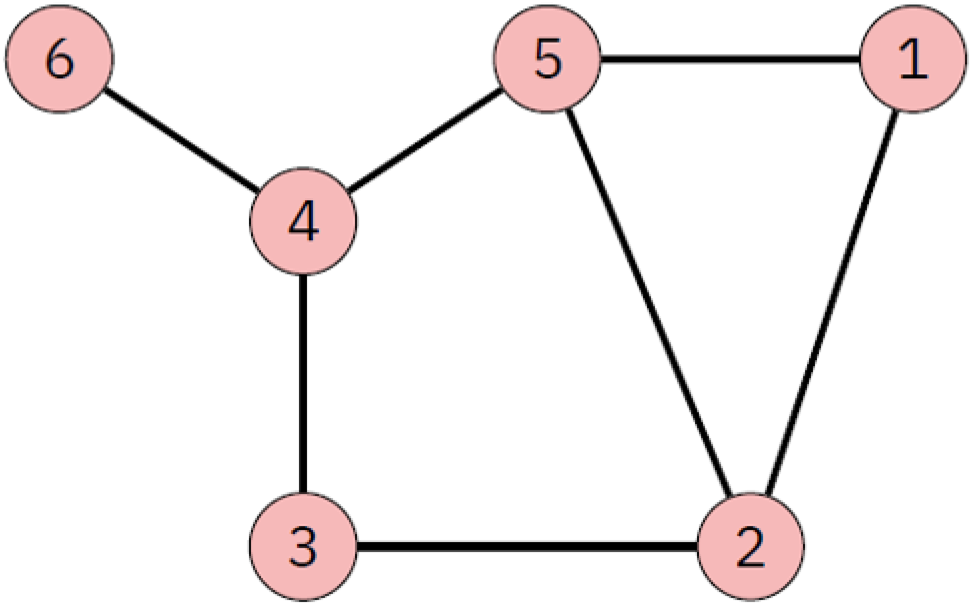

$$Radius(\{1,2,5\}) = ?$$

Further Definitions

- Let S be a part of the graph (possibly whole graph)
- The radius of S is defined as

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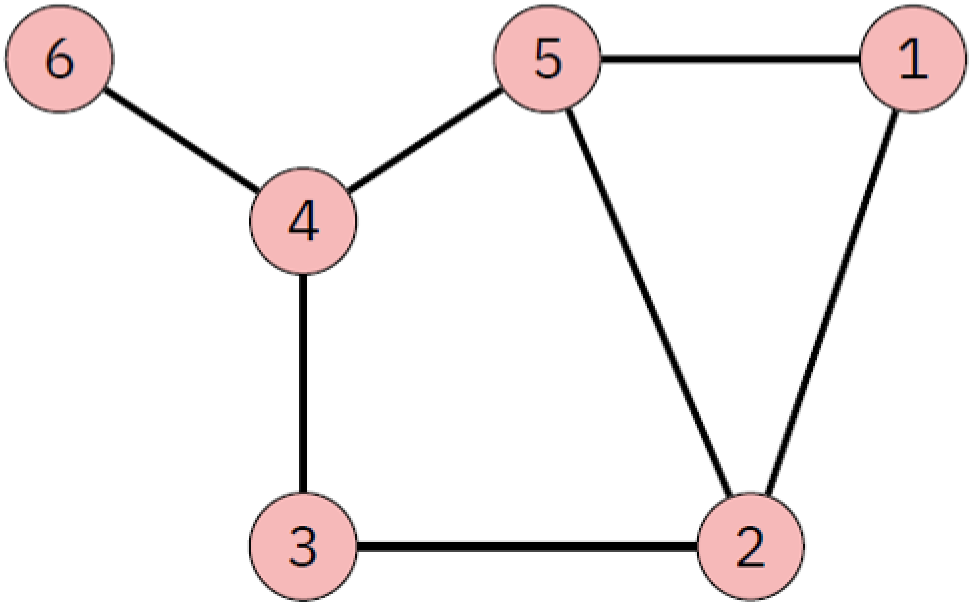

$$Radius(\{1,2,5\}) = ?$$

$Ecc(1, \{1,2,5\})$

$Ecc(2, \{1,2,5\})$

$Ecc(5, \{1,2,5\})$

Further Definitions

- Let S be a part of the graph (possibly whole graph)
- The radius of S is defined as

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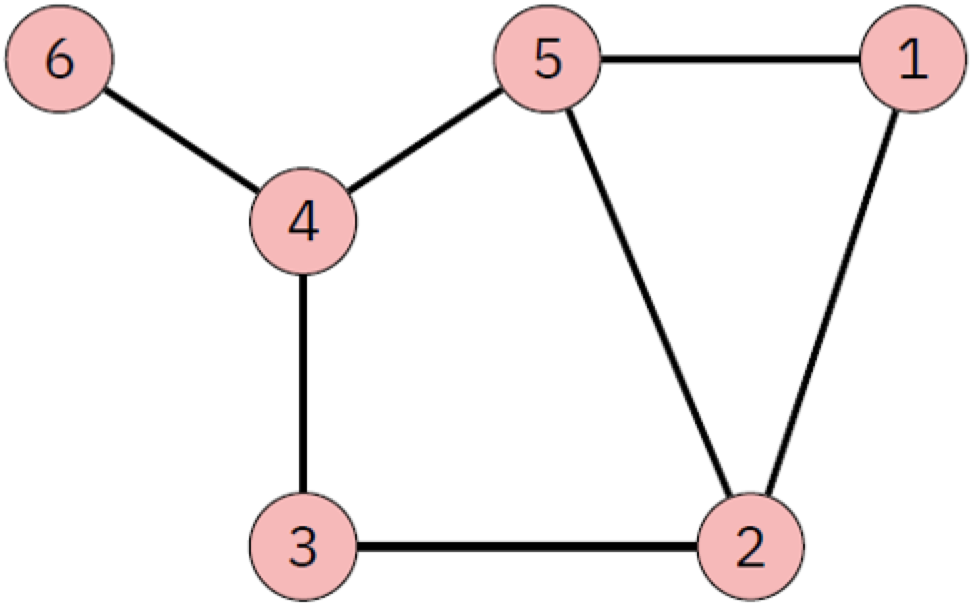


$$Radius(\{1,2,5\}) = ?$$

$$Ecc(1, \{1,2,5\}) = \max\{Dist(1,1), Dist(1,2), Dist(1,5)\}$$

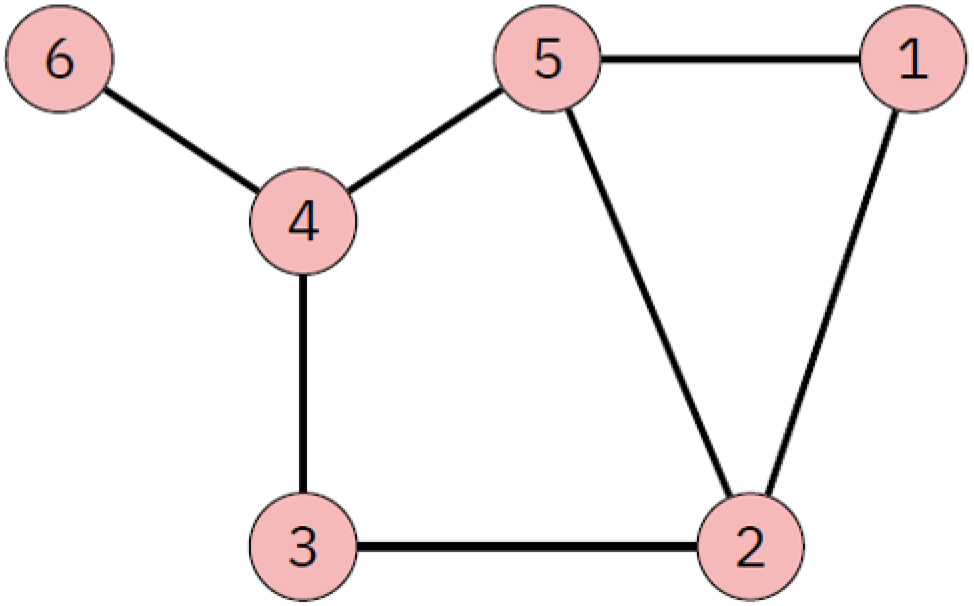
$$Ecc(2, \{1,2,5\}) = \max\{Dist(2,1), Dist(2,2), Dist(2,5)\}$$

$$Ecc(5, \{1,2,5\}) = \max\{Dist(5,1), Dist(5,2), Dist(5,5)\}$$

Further Definitions

- Let S be a part of the graph (possibly whole graph)
- The radius of S is defined as

$$\text{Radius}(S) = \min_{i \in S} \text{Ecc}(i, S)$$



$$Radius(\{1,2,5\}) = ?$$

$$Ecc(1, \{1,2,5\}) = \max\{Dist(1,1), Dist(1,2), Dist(1,5)\} = 1$$

$$Ecc(2, \{1,2,5\}) = \max\{Dist(2,1), Dist(2,2), Dist(2,5)\} = 1$$

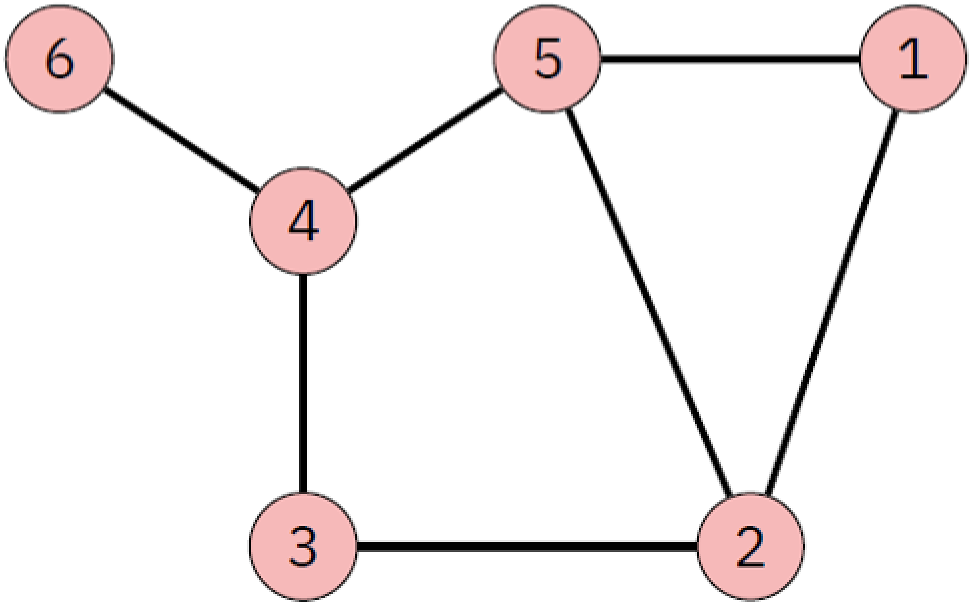
$$Ecc(5, \{1,2,5\}) = \max\{Dist(5,1), Dist(5,2), Dist(5,5)\} = 1$$

$$Radius(\{1,2,5\}) = \min\{1,1,1\} = 1$$

Further Definitions

- Let S be a part of the graph (possibly whole graph)
- The radius of S is defined as

$$\boxed{\text{Radius}(S) = \min_{i \in S} \text{Ecc}(i, S)}$$



$$Radius(\{1,2,5\}) = 1$$

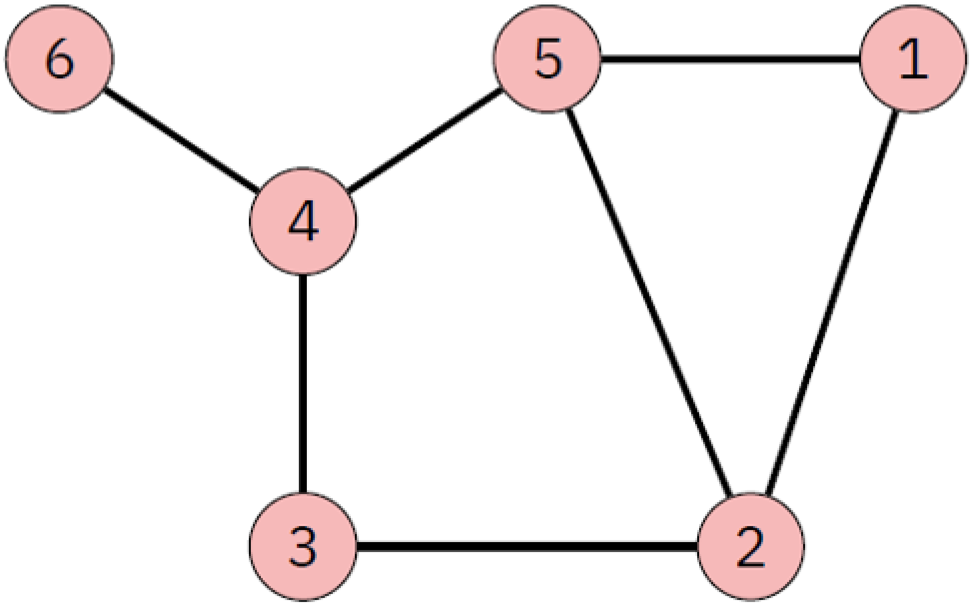
$$Radius(\{3,5,6\}) = ?$$

$$Radius(G) = ?$$

Further Definitions

- Let S be a part of the graph (possibly whole graph)
- The radius of S is defined as

$$\boxed{\text{Radius}(S) = \min_{i \in S} \text{Ecc}(i, S)}$$



$$Radius(\{1,2,5\}) = 1$$

$$Radius(\{3,5,6\}) = 2$$

$$Radius(G) = 2$$

Further Definitions

- Let S be a part of the graph (possibly whole graph)
- The radius of S is defined as

$$\text{Radius}(S) = \min_{i \in S} \text{Ecc}(i, S)$$

A vertex v is a center of S if $Ecc(v, S) = Radius(S)$

Further Definitions

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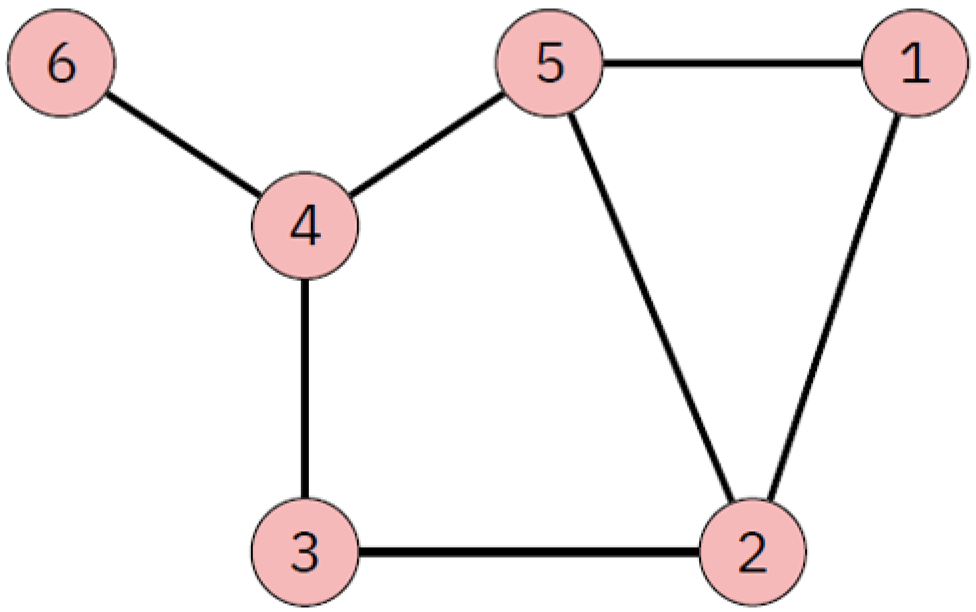

A vertex v is a center of S if
$Ecc(v, S) = Radius(S)$

$$Radius(G) = 2$$

Further Definitions

- Let S be a part of the graph (possibly whole graph)
- The radius of S is defined as

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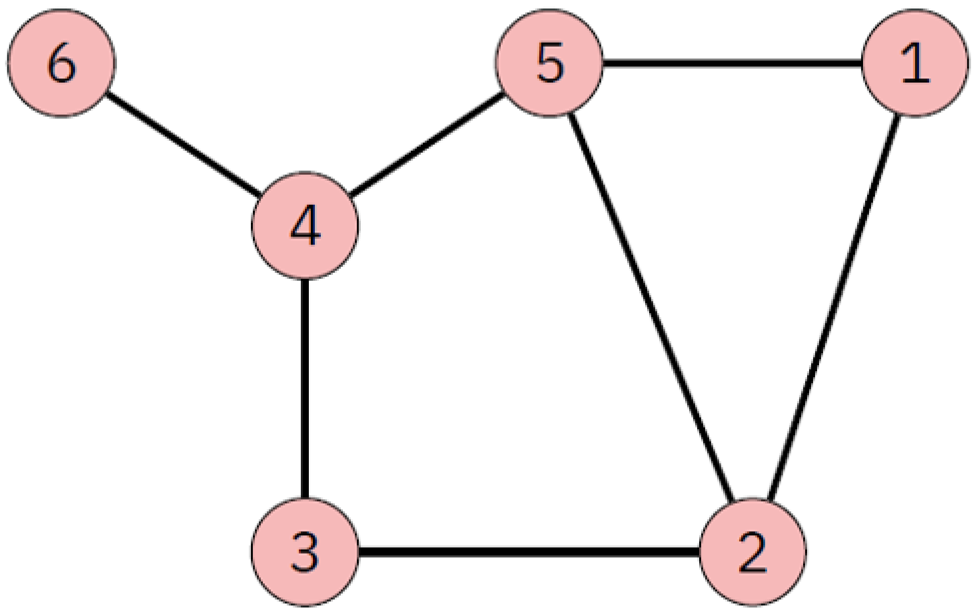


A vertex v is a center of S if
$Ecc(v, S) = Radius(S)$

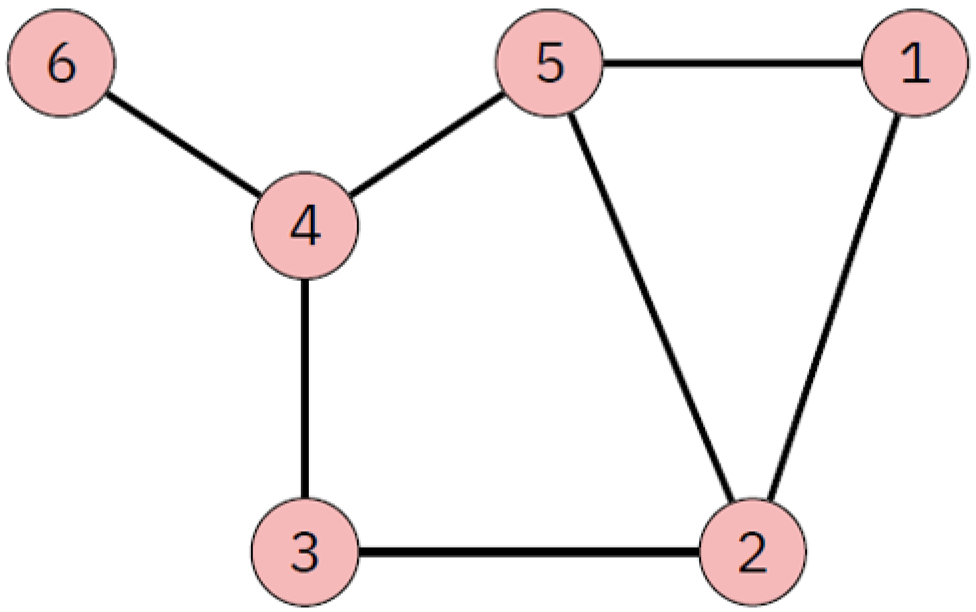
$$Radius(G) = 2$$

4 is a center of G since $Ecc(4, G) = 2$

Further Definitions

- Let S be a part of the graph (possibly whole graph)
- The radius of S is defined as

$$\text{Radius}(S) = \min_{i \in S} \text{Ecc}(i, S)$$



A vertex v is a center of S if $Ecc(v, S) = Radius(S)$

- There can be more than one center!
- Each of 1,2,5 is a center of $\{1,2,5\}$

EXERCISE 7. The diameter of a circle is always twice the radius. Does this also hold for graphs? Experiment with some simple graphs. You may assume that the graph is connected.

EXERCISE 7. The diameter of a circle is always twice the radius. Does this also hold for graphs? Experiment with some simple graphs. You may assume that the graph is connected.

$Diameter(G) \leq 2 \cdot Radius(G)$ always holds!