Synchronization on complex networks

A model for neural networks

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26 March 2021

What will I tell you today?

- a short introduction

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- a short introduction
- what a stochastic process is

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- Kuramoto: a mathematical model
- when there is synchronization

Processes on networks

Introduction

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2. Searching for information on the internet

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3. Continuous space or discrete space
4. Dynamic or static network

Examples of synchronization

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If you are looking for your next popular science book to read try: 'Sync: The emerging science of spontaneous order' - Steven Strogatz

YouTube Video

Stochastic processes

Question: What do you think?

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Question: What do you think?

Ingredients

- some randomness

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Example: Coin flipping

- win 1 € if heads
- lose 1 € if tails

$$\omega = \{H, T, T, T, T, H, T, H \dots\}$$

Exercise

How can you calculate your wealth after n coin flips?

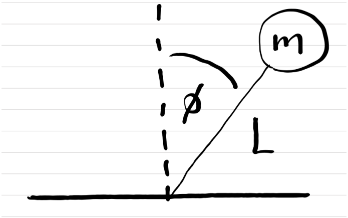
1. Given a sequence of coin-flip outcomes ω construct a function $X_n(\omega)$ representing your wealth after the n^{th} flip, if you win a euro when the outcome is heads and lose a euro if the outcome is tails.
2. Calculate the probability of having three euros after five coin flips given that the coin is head with probability $1/2$.

Solution:

1. $X_n(\omega) = \sum_{i=1}^{n} \left[\mathbb{1}_{\{\omega_i = H\}} - \mathbb{1}_{\{\omega_i = T\}} \right]$
2. $\mathbb{P}[X_5(\omega) = 4] = \binom{5}{4}(1/2)^3(1/2)^2 = 5/32.$

Single noisy oscillator

Consider a metronome in the wind:



"State" characterized by position and velocity. Can we go simpler? Consider a circle:

- all the way to the right $\rightarrow$ top point (0 degrees)
- straight and moving to the left $\rightarrow$ right point ($\pi/2$ degrees)
- all the way to the left $\rightarrow$ bottom point (π degrees)
- straight and moving to the right $\rightarrow$ left point ($3\pi/2$ degrees)

Single noisy oscillator

So each position of the metronome can be described by an angle $\theta \in [0, 2\pi)$. Depending on the mass (m) and length L it will move with some frequency, say ω. We can then write the evolution of the metronome as:

$$\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \omega + \xi(t) \tag{1}$$

$\xi(t)$ is noise which has $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2D\delta(t - t')$.

- $\xi(t)$ is constructed using the sort of procedure we went through earlier
- the difference here is that time is a continuous "index".
- $\xi(t)$ is representing the random aspect of the evolution due to the wind.

Online Animation

To illustrate please open up:

`https://www.networkpages.nl/synchrony-animation/`

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And fill in: $N_1 = 1$, $N_2 = 0$, $K_1 = L_1 = K_2 = L_2 = 0$, $D = 1$ under the Dynamics section. In the Initialization section set both $u_1 = u_2 = 0$ and $r_1 = r_2 = 0$ and pick between "Constant zero" and "Standard normal" for your frequency distribution.

Q: What happens?

Including an interaction (two metronomes)

What happens if I include the following interaction?

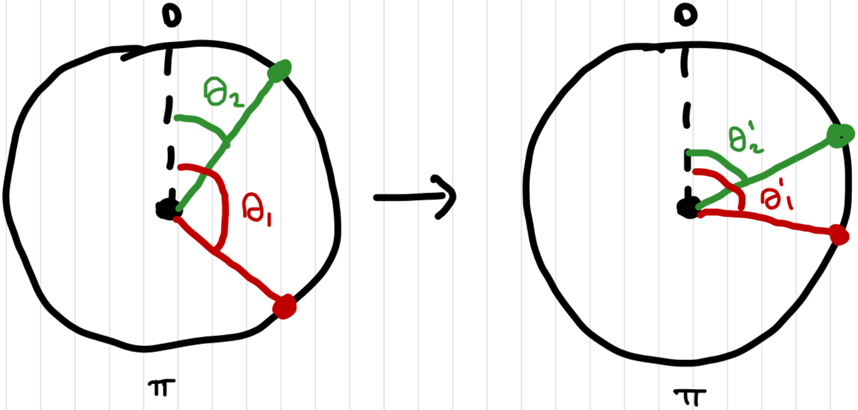
$$\frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} = \omega_1 + \sin(\theta_2 - \theta_1) + \xi_1(t) \tag{2}$$

$$\frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} = \omega_2 + \sin(\theta_1 - \theta_2) + \xi_2(t) \tag{3}$$

- Consider two points on the circle.
- Determine the sign of the interaction terms.
- What can you conclude?

[Exercise and poll]

Solution



$$\sin(\theta_1 - \theta_2) > 0 \,, \quad \sin(\theta_2 - \theta_1) < 0$$

Break

[10 minute break!]

(Noisy) Kuramoto model

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Achtung! Mathematics ahead!

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- $\theta_i(t)$ – phase of i^{th} oscillator

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Oscillators evolve according to a system of coupled stochastic differential equations

$$\frac{\mathrm{d}\theta_i(t)}{\mathrm{d}t} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin\left[\theta_j(t) - \theta_i(t)\right] + \xi_i(t). \tag{4}$$

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Here, $K \in (0, \infty)$ is the interaction strength, $D \in (0, \infty)$ is the noise strength (contained in $\xi_i(t)$).

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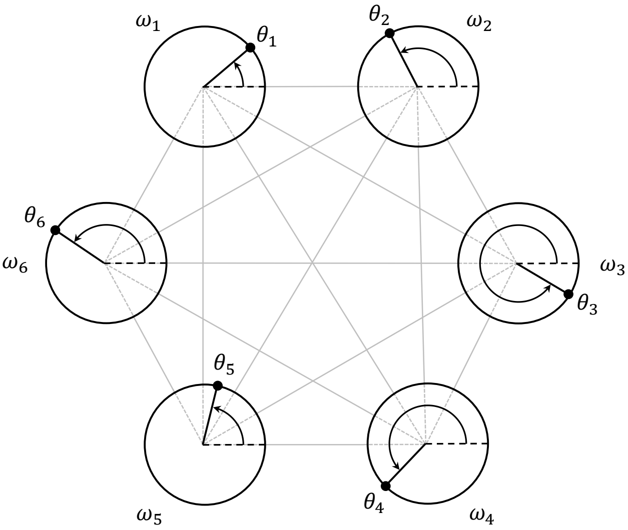
Question:

Can you spot the network here?

Cartoon of the Kuramoto model for $N = 6$



Keeping track of the order

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Order parameter

$$r_N(t)\, \mathrm{e}^{i\psi_N(t)} = \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{i\theta_j(t)}. \tag{5}$$

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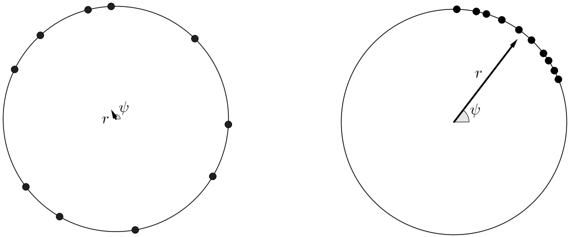

Phase distributions with $r = 0.095$ and $r = 0.929$.

Kuramoto's Trick

- Multiplying both sides by $\mathrm{e}^{-i\theta_i}$ gives

$$r_N(t)\mathrm{e}^{i(\psi_N(t)-\theta_i(t))} = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{i(\theta_j(t)-\theta_i(t))}. \tag{6}$$

- Express both complex exponentials using Euler

$$r_N(t)\cos(\psi_N(t)-\theta_i(t)) + i r_N(t)\sin(\psi_N(t)-\theta_i(t)) \tag{7}$$

$$= \frac{1}{N}\sum_{j=1}^{N}\Big[\cos(\theta_j(t)-\theta_i(t)) + i\sin(\theta_j(t)-\theta_i(t))\Big].$$

- Rewrite:

$$r_N(t)\cos(\psi_N(t)-\theta_i(t)) = \frac{1}{N}\sum_{j=1}^{N}\cos(\theta_j(t)-\theta_i(t))$$

$$r_N(t)\sin(\psi_N(t)-\theta_i(t)) = \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j(t)-\theta_i(t)).$$

Kuramoto's Trick

Plugging in to the evolution equation gives

$$\mathrm{d}\theta_i(t) = K\, r_N(t) \sin\left[\psi_N(t) - \theta_i(t)\right] \mathrm{d}t + D\, \mathrm{d}W_i(t), \qquad (8)$$

- The equation basically says that you only interact with the *average* angle of the oscillators
- and that your interaction is *modulated* by the amount of synchronization there is.
- This is an example of what is called a *mean-field* model.

Notice the importance of the choice of order parameter!

Adding more oscillators and waiting

The large N limit

As N gets ever larger, you can describe the evolution of the oscillators as the evolution of a density.

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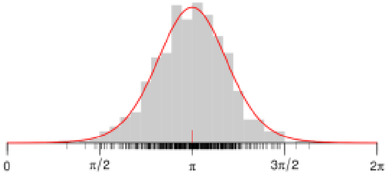
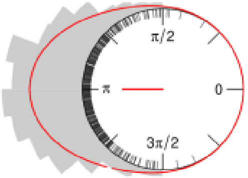
The large time limit (steady-state)

Question: Does the density of the system stop evolving at some point?

Density





Break

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Feel free to experiment with the online animation.

Critical threshold

There exists a critical threshold K_c such that:

(I) For $K < K_c$ the system relaxes to an unsynchronized state $(r = 0)$.

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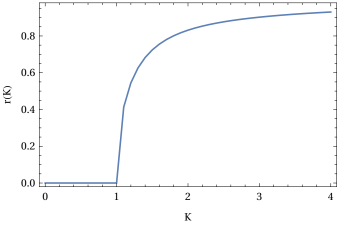
Theorem

If $D = 1$, then $K_c = 1$.



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- What happens?

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- What happens?
- What happens if you set $K_1 = 1$? Does that contradict the previous result?

What if $D \neq 1$?

For steady-state solution r must satisfy:

$$r = V\left(\frac{2Kr}{D}\right), \tag{9}$$

where $V(\cdot)$ is a special function with $V(0) = 0$ and $V'(0) = 1/2$. For critical threshold use *Taylor expansion*, i.e., for small values of x

$$V(x) < V(0) + V'(0)\, x. \tag{10}$$

- What is the critical threshold now?
- Does this make sense?

[Exercise and poll]

Solution

- By using the function properties in (10) we get

$$V(x) < \frac{1}{2}\,x. \tag{11}$$

- Replace the right hand side of (9) with this equation so

$$r < \frac{1}{2}\,\frac{2Kr}{D}. \tag{12}$$

Canceling the r's, 2's and moving everything except the K to the other side we obtain the new critical threshold:

$$K_c = D < K. \tag{13}$$

- Yes! If there is more noise in the system (corresponding to a larger value of D) we expect that it will be harder to synchronize so that we will need a larger K.

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- Using mathematics we can determine a critical threshold.
- Synchronization is achieved if the interaction wins the battle against the noise.

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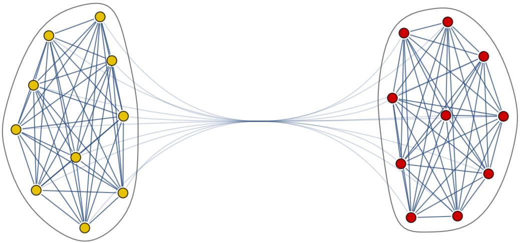

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