University of Amsterdam

# Finding the optimal route in a road traffic network

## GRAPH THEORY

Networks goes to school
March 17, 2021

Rens Kamphuis

NETWORKS
THENETWORKCENTER.NL

# What are networks?

## network

***noun*** [ C ]

**UK** /ˈnet.wɜːk/ **US** /ˈnet.wɝːk/

**B2**

**a large system consisting of many similar parts that are connected together to allow movement or communication between or along the parts**

# What are networks?


These are all networks!

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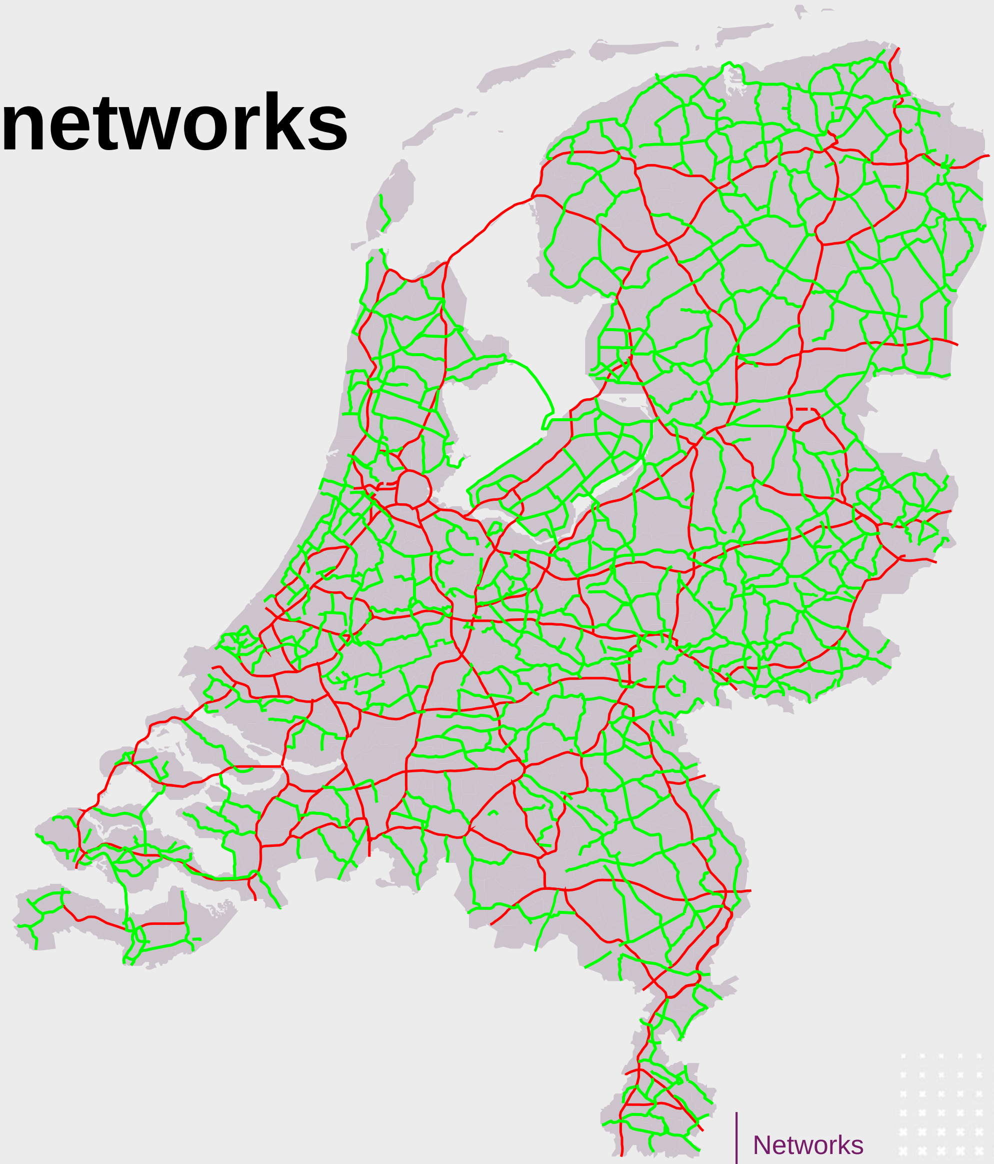
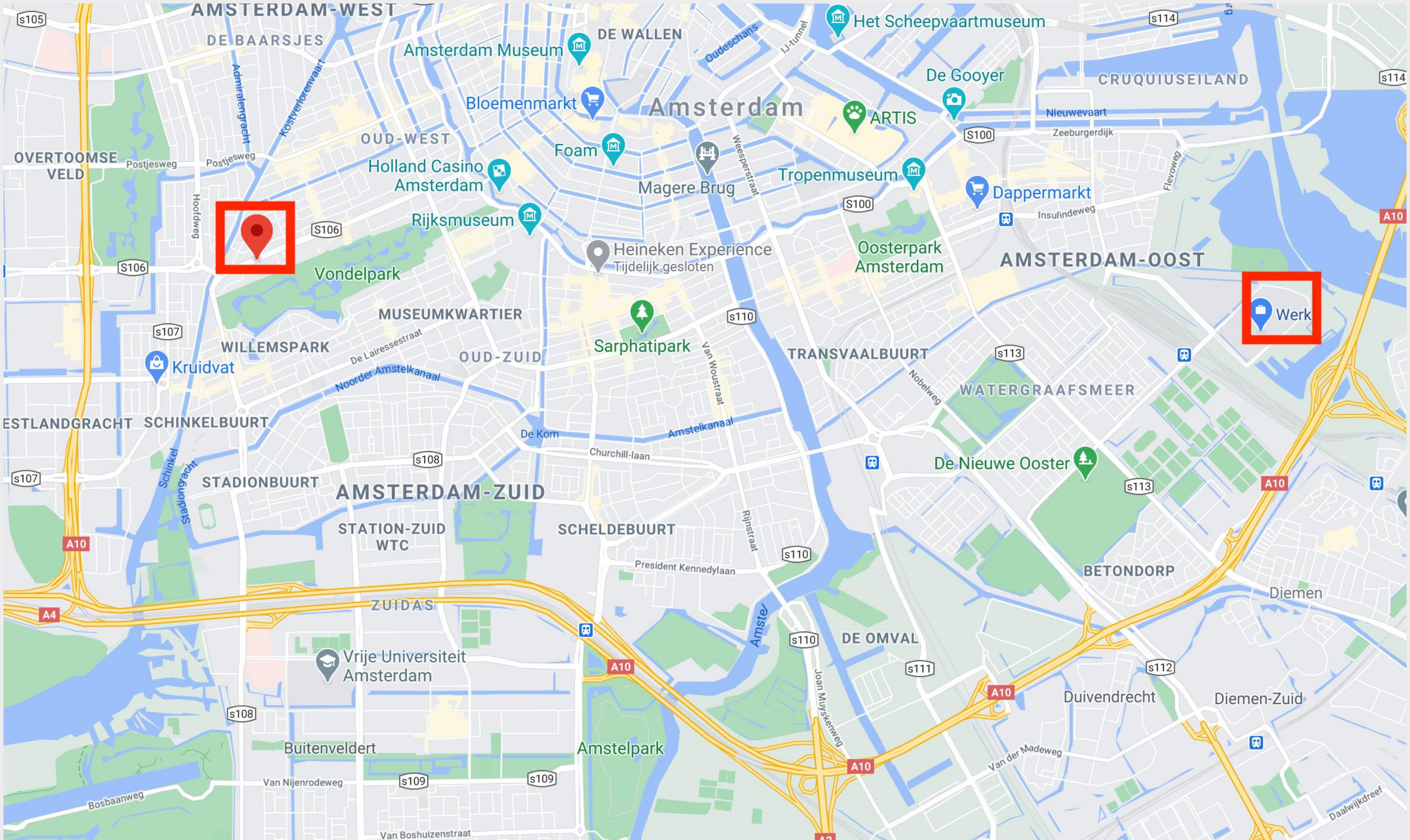
# Road networks

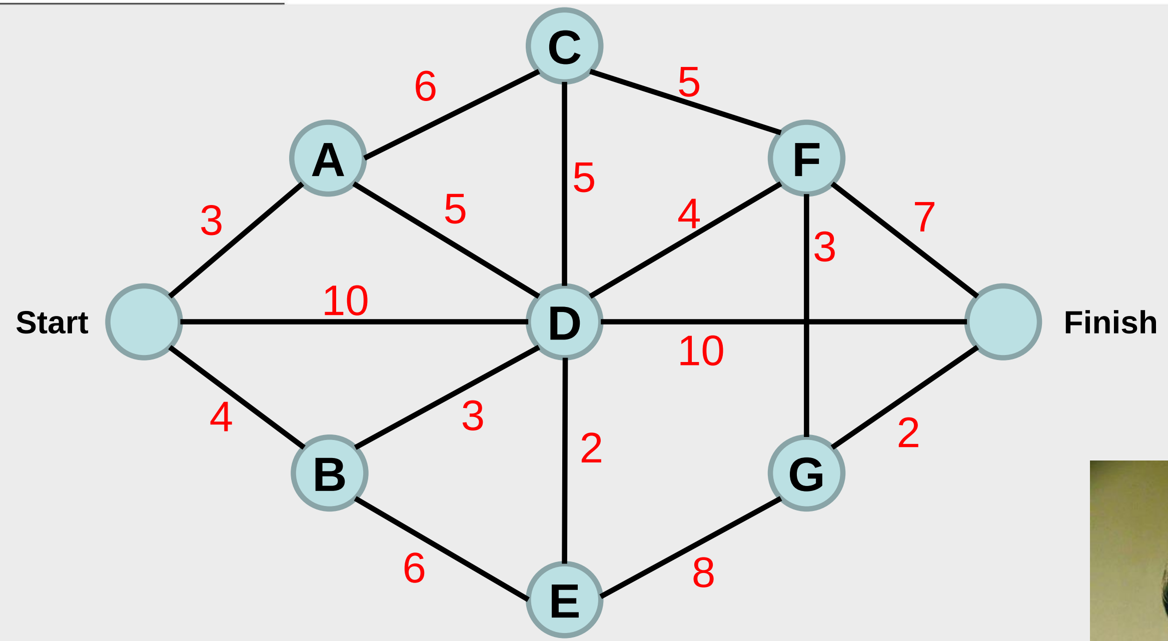


# Why study networks?

- What is the shortest route to reach my destination?
- What route is least likely to suffer from traffic jams?
- What route should I take to arrive at my destination *in time*?

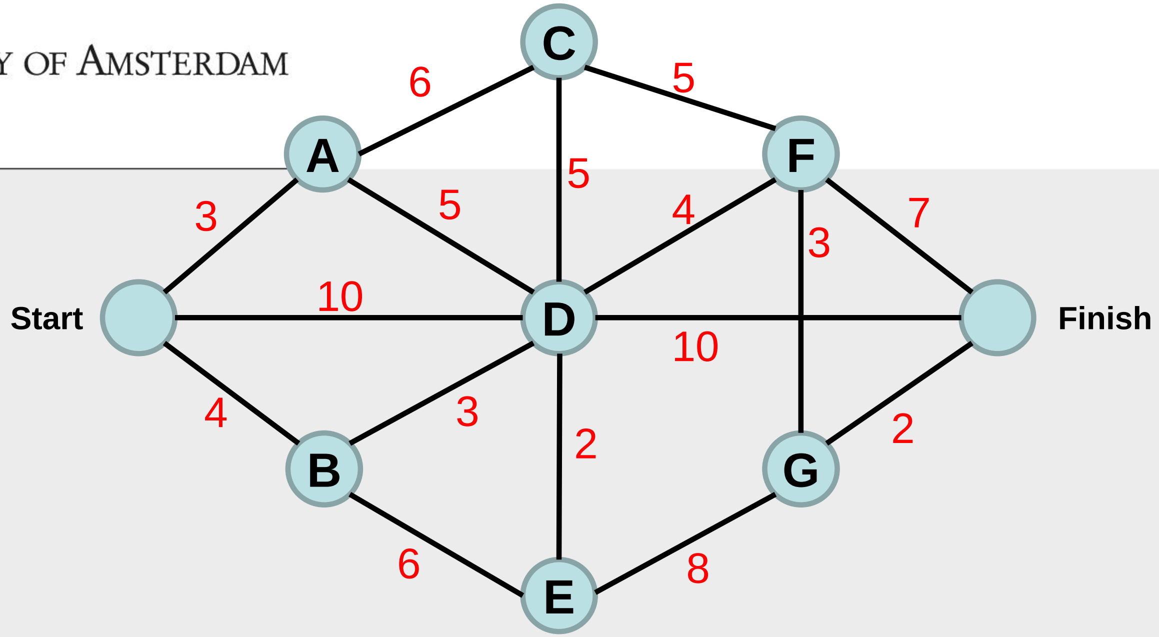
# Finding the shortest path can be difficult






- What is the shortest route from start to finish?
- Dijkstra's algorithm can be used to determine the shortest route in a network

Edsger W. Dijkstra (1930-2002)

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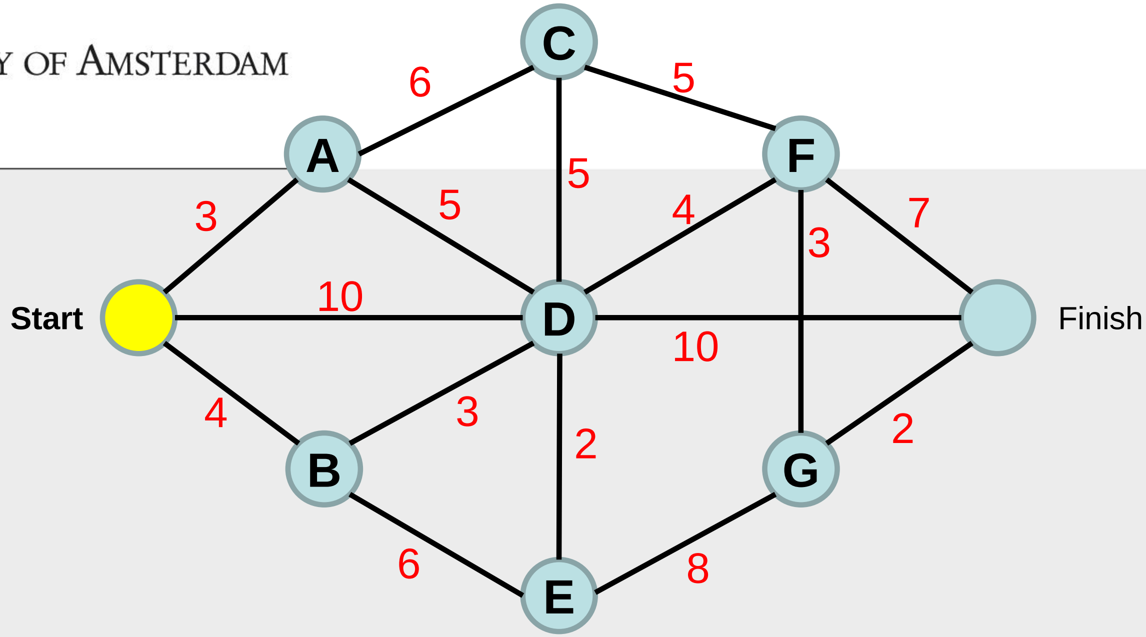


| A | B | C | D | E | F | G | Finish |
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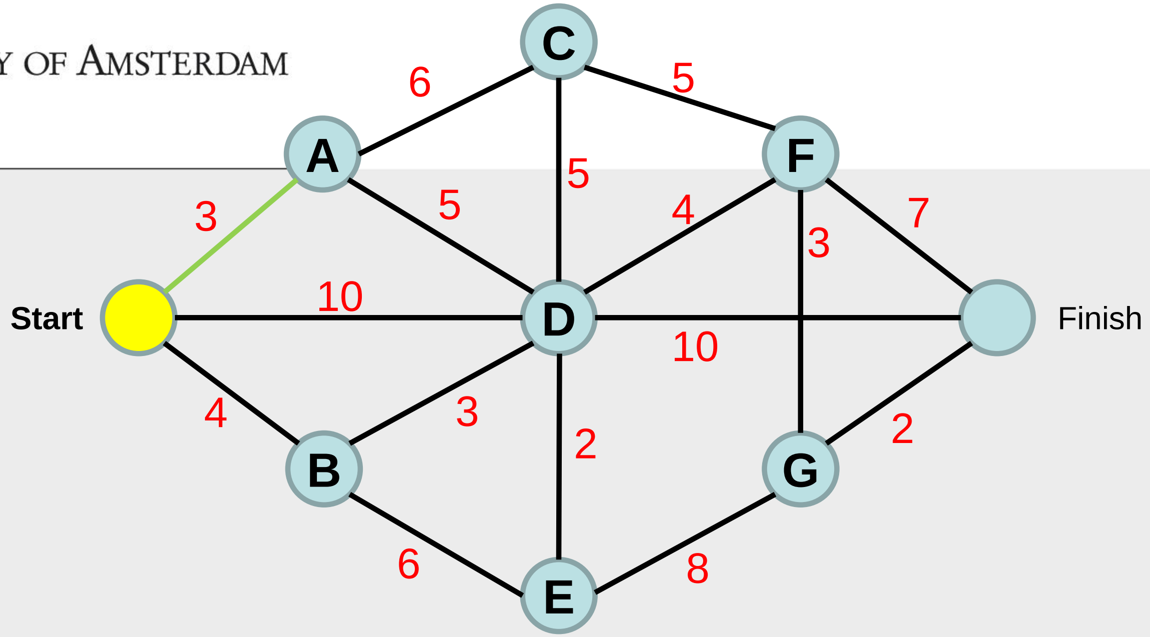


| A | B | C | D | E | F | G | Finish |
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| **3, start** | 4, start | | 10, start | | | | |
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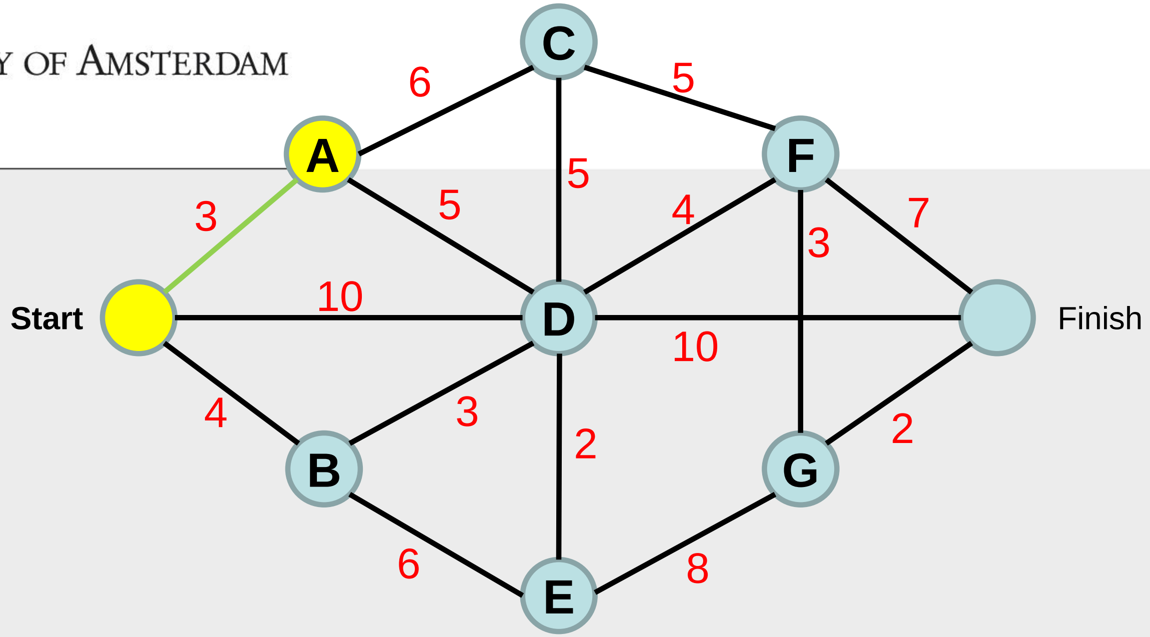


| A | B | C | D | E | F | G | Finish |
|---|---|---|---|---|---|---|---|
| **3, start** | 4, start | | 10, start | | | | |
| * | 4, start | 9, A | 8, A | | | | |
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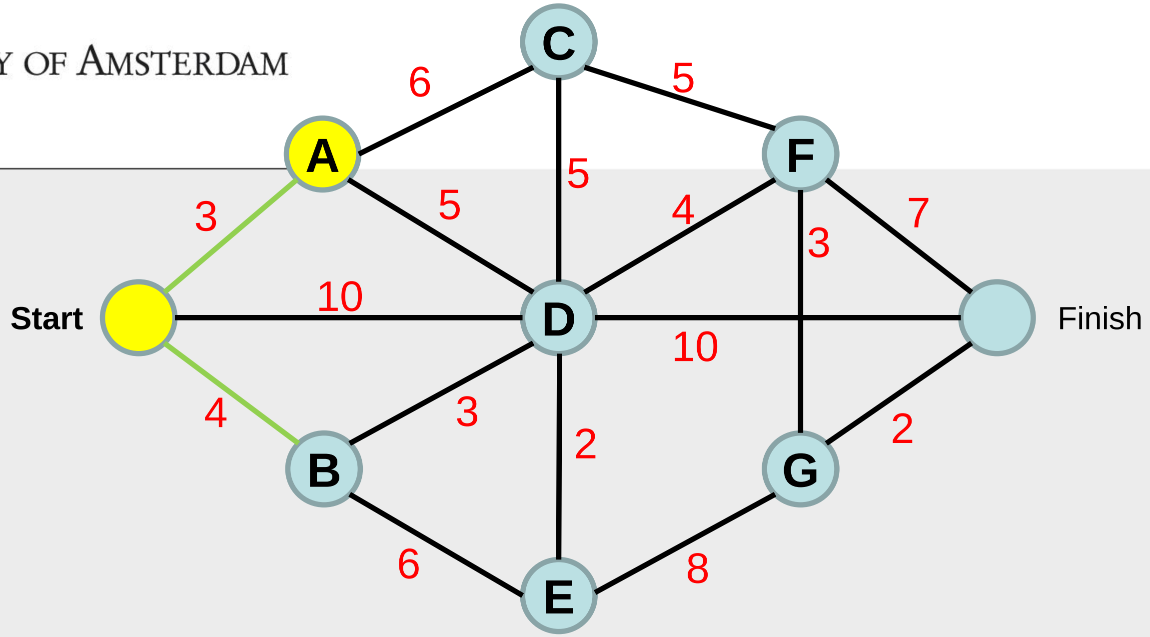


| A | B | C | D | E | F | G | Finish |
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| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
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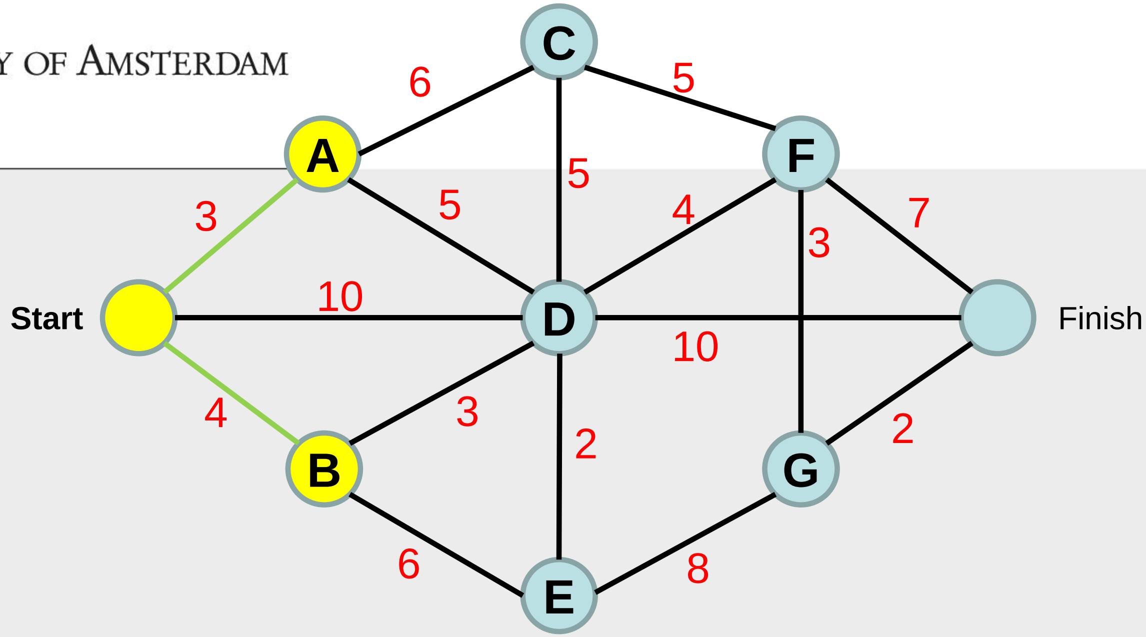


| A | B | C | D | E | F | G | Finish |
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| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
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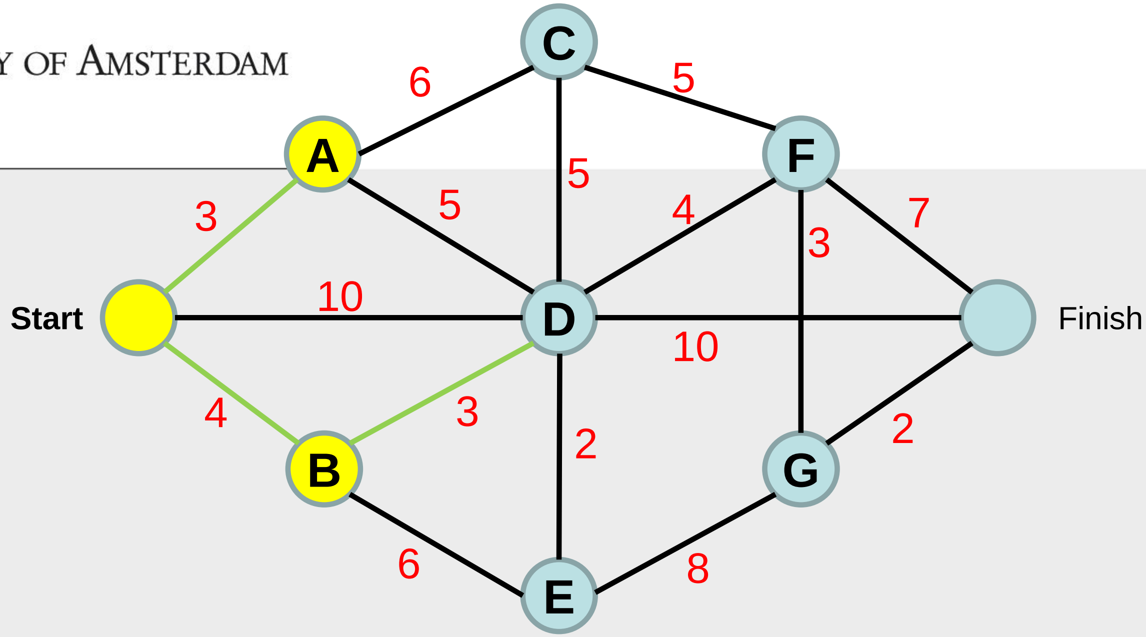


| A | B | C | D | E | F | G | Finish |
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| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
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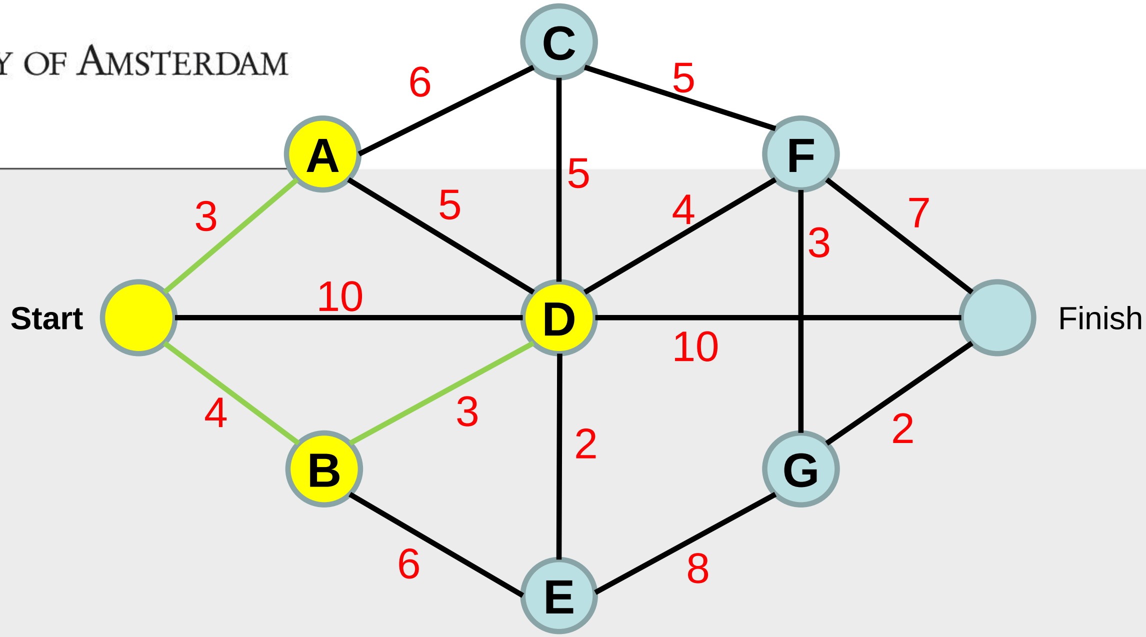


| A | B | C | D | E | F | G | Finish |
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| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
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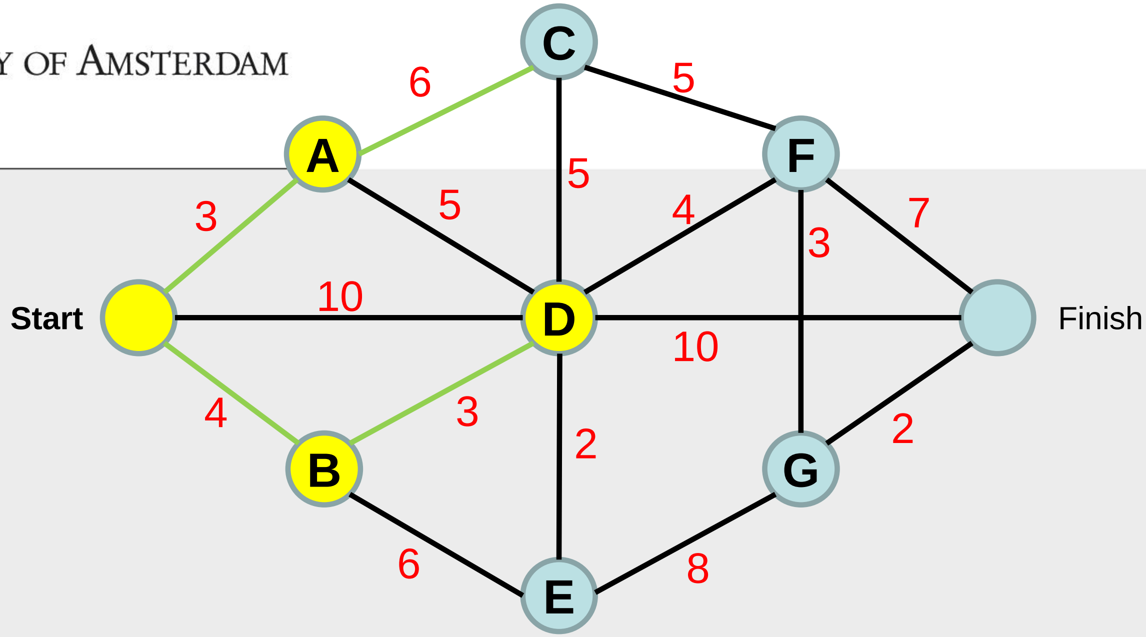


| A | B | C | D | E | F | G | Finish |
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| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
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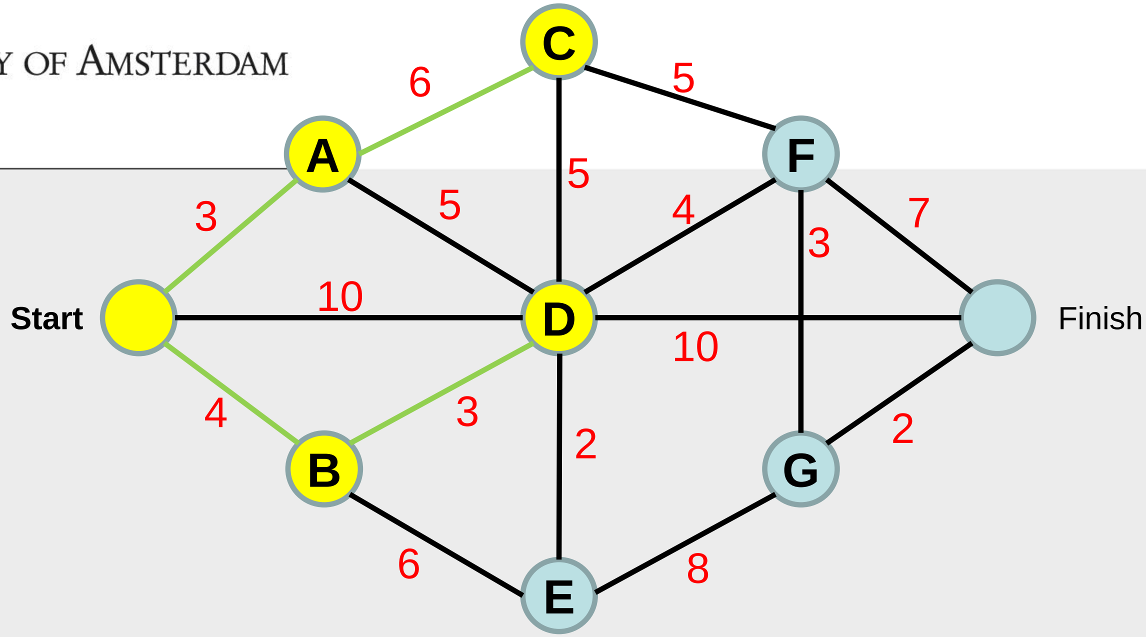


| A | B | C | D | E | F | G | Finish |
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| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | 9, D | 11, D | | 17, D |
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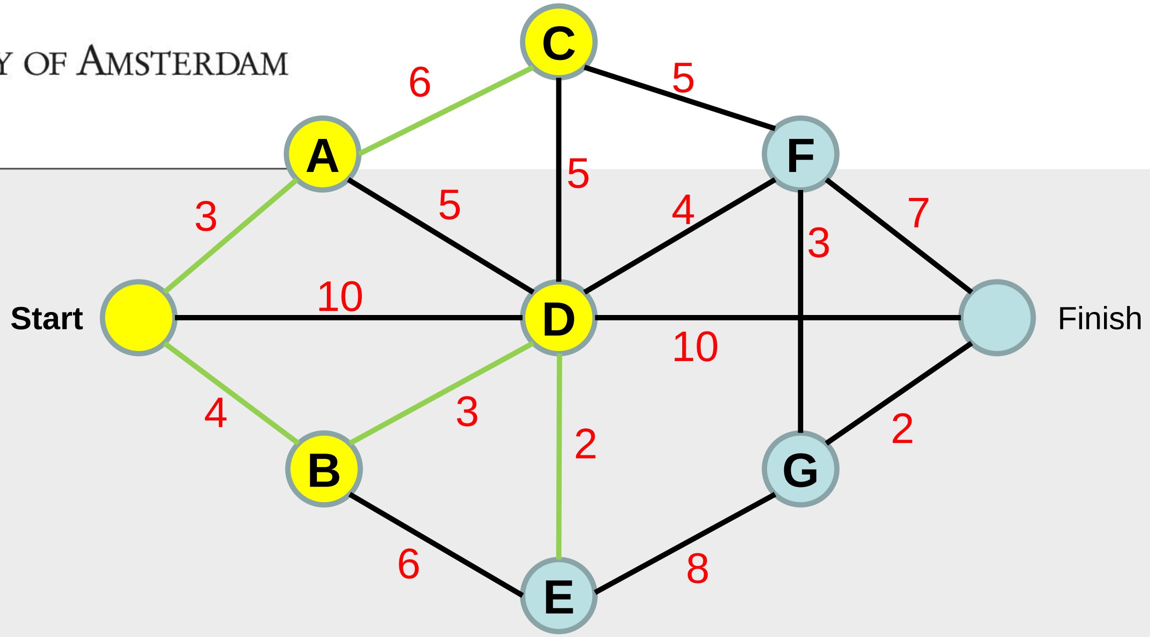


| A | B | C | D | E | F | G | Finish |
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| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | **9, D** | 11, D | | 17, D |
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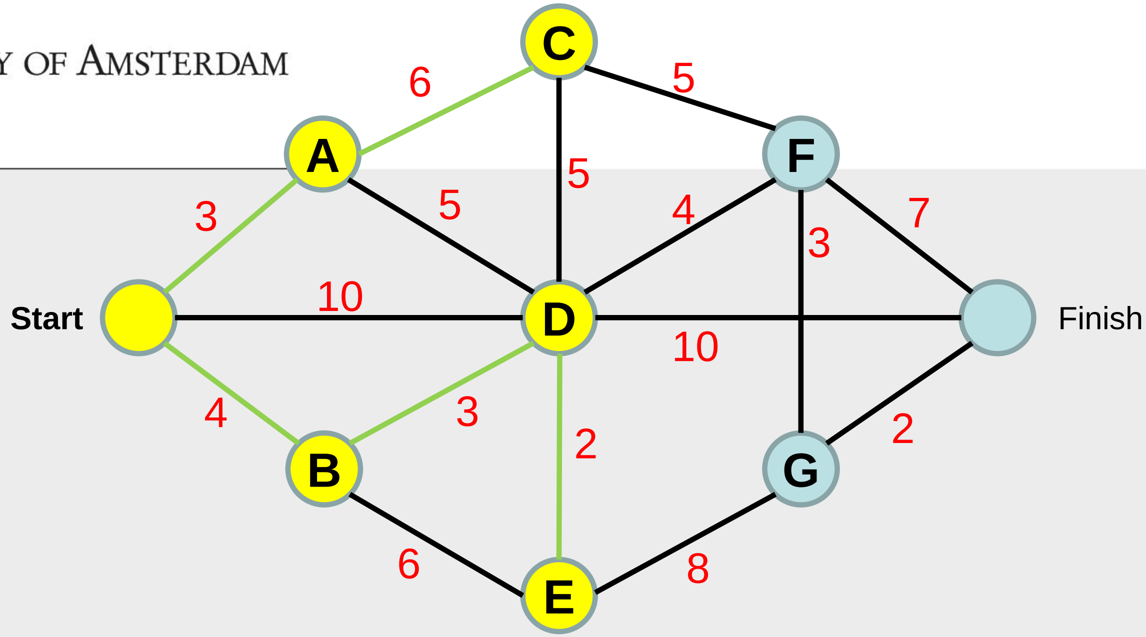


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| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | **9, D** | 11, D | | 17, D |
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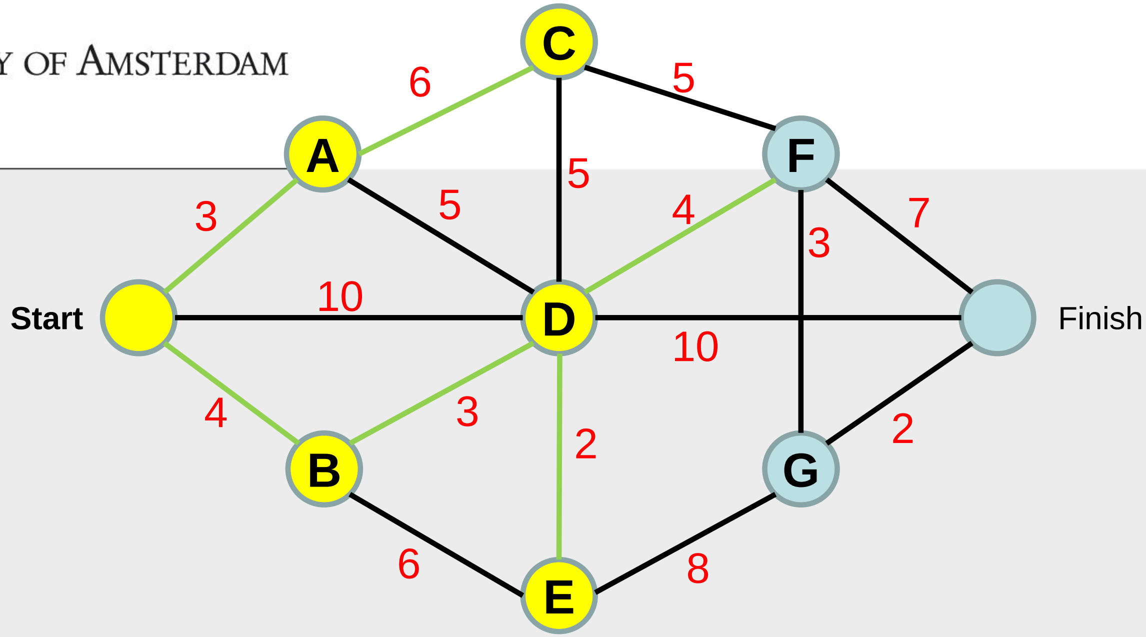


| A | B | C | D | E | F | G | Finish |
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| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | **9, D** | 11, D | | 17, D |
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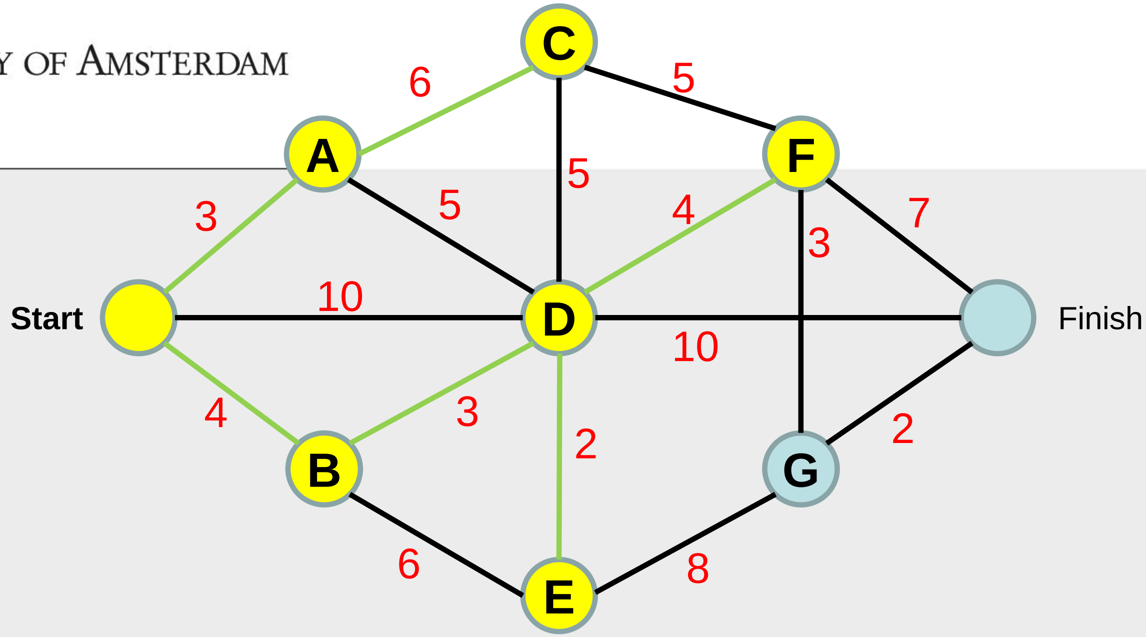


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| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | **9, D** | 11, D | | 17, D |
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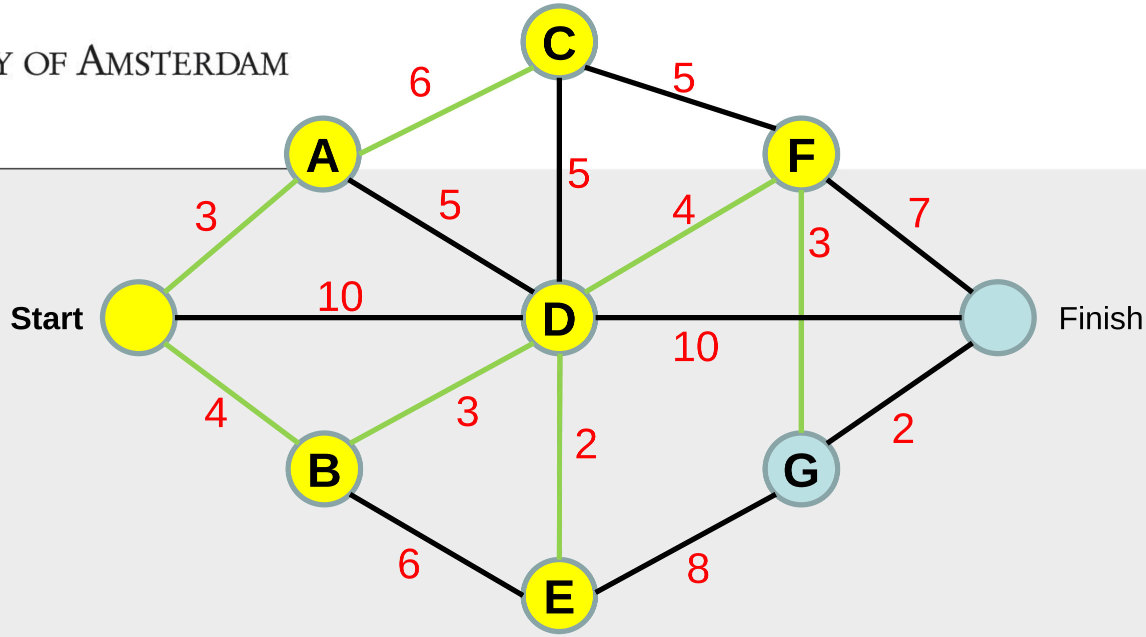


| A | B | C | D | E | F | G | Finish |
|---|---|---|---|---|---|---|---|
| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | **9, D** | 11, D | | 17, D |
| * | * | * | * | * | **11, D** | 17, E | 17, D |
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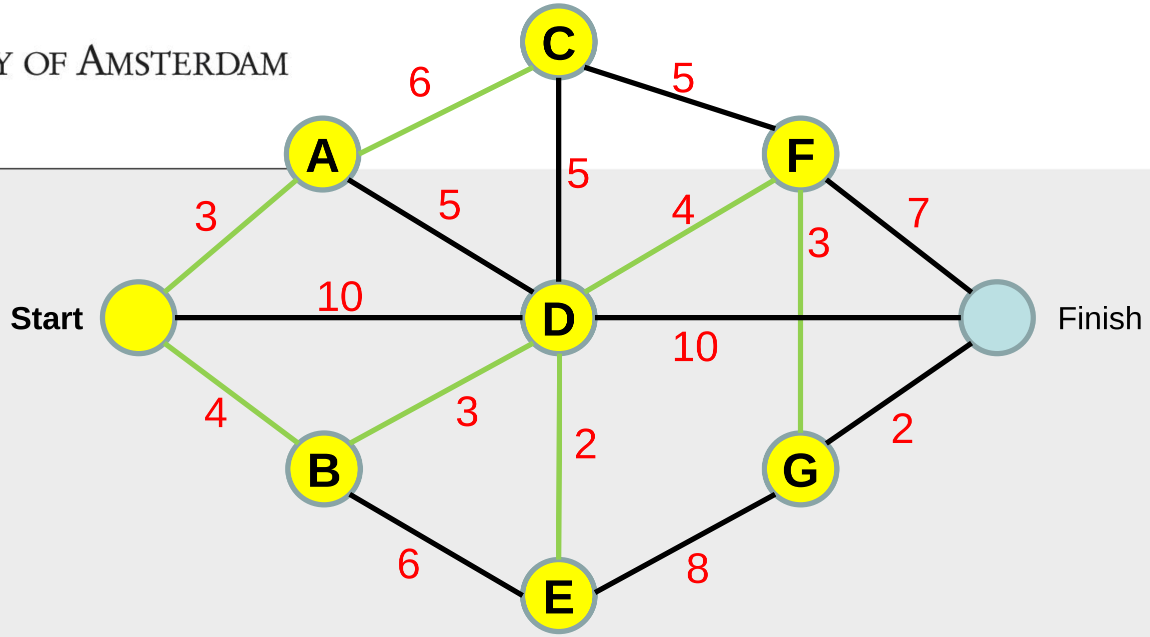


| A | B | C | D | E | F | G | Finish |
|---|---|---|---|---|---|---|---|
| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | **9, D** | 11, D | | 17, D |
| * | * | * | * | * | **11, D** | 17, E | 17, D |
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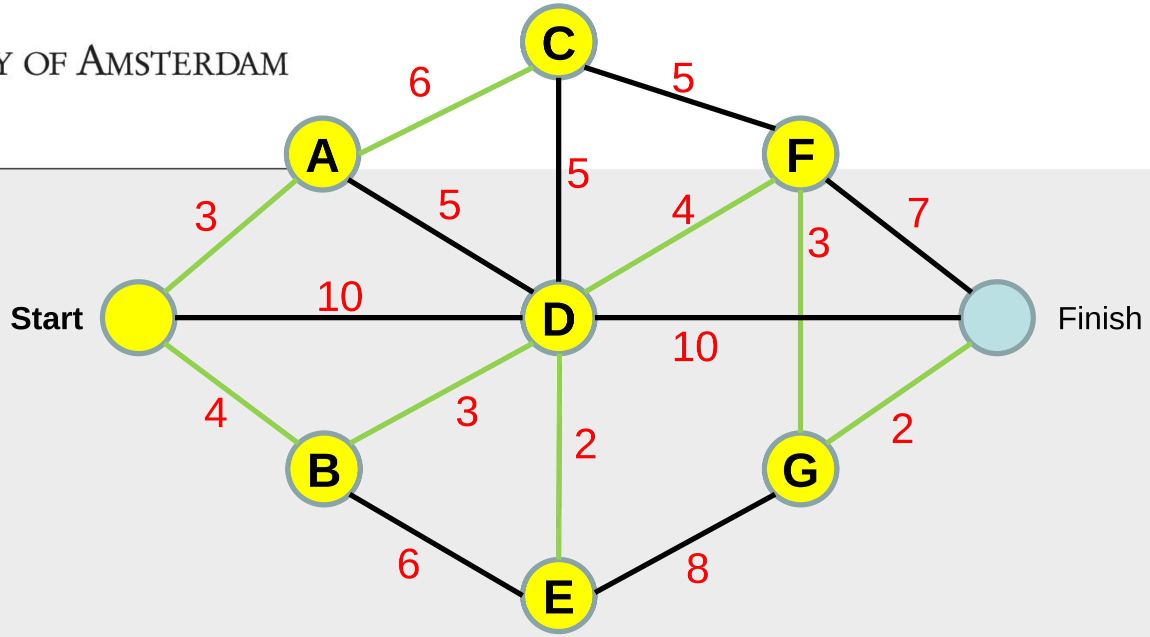
| * | * | * | * | * | * | * | 16, G |




| A | B | C | D | E | F | G | Finish |
|---|---|---|---|---|---|---|---|
| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | **9, D** | 11, D | | 17, D |
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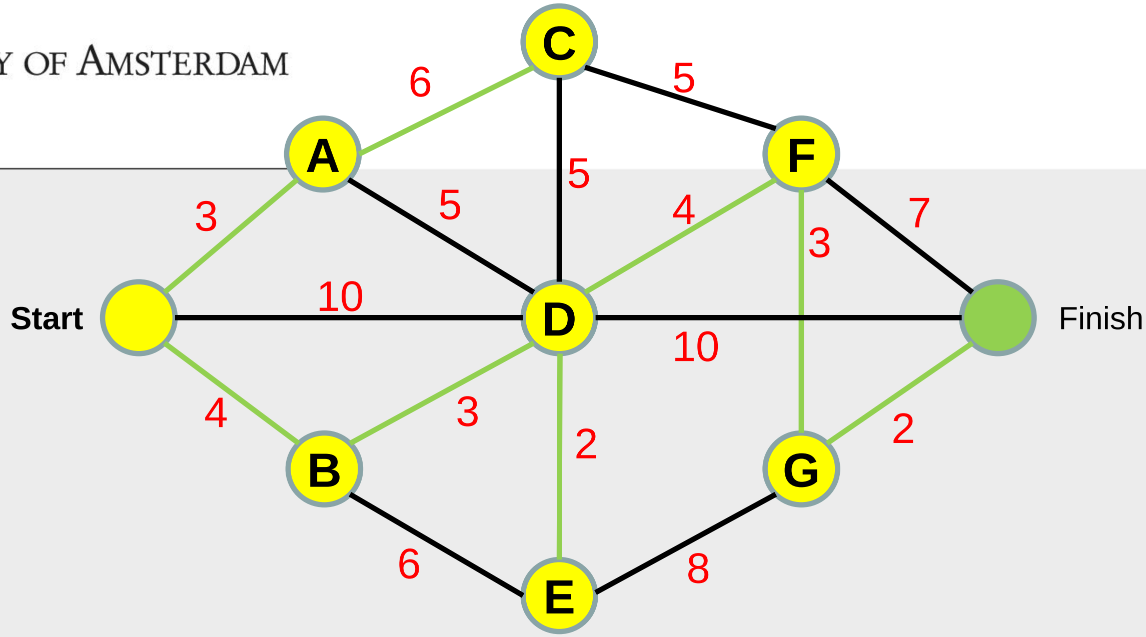


| A | B | C | D | E | F | G | Finish |
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| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | **9, D** | 11, D | | 17, D |
| * | * | * | * | * | **11, D** | 17, E | 17, D |
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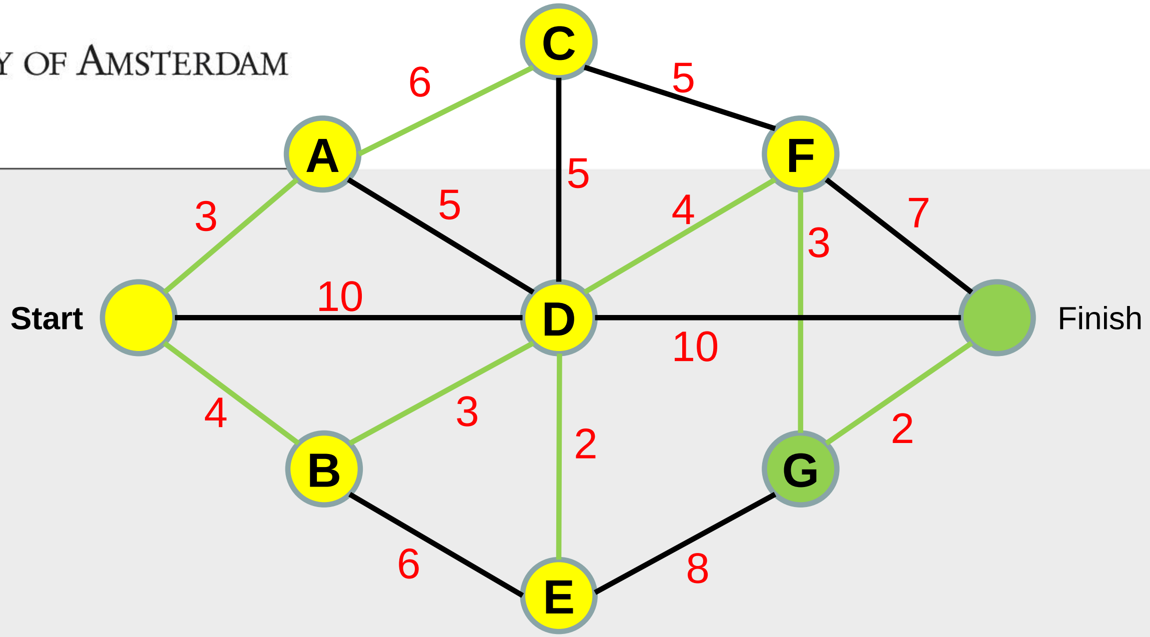


| A | B | C | D | E | F | G | Finish |
|---|---|---|---|---|---|---|---|
| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | **9, D** | 11, D | | 17, D |
| * | * | * | * | * | **11, D** | 17, E | 17, D |
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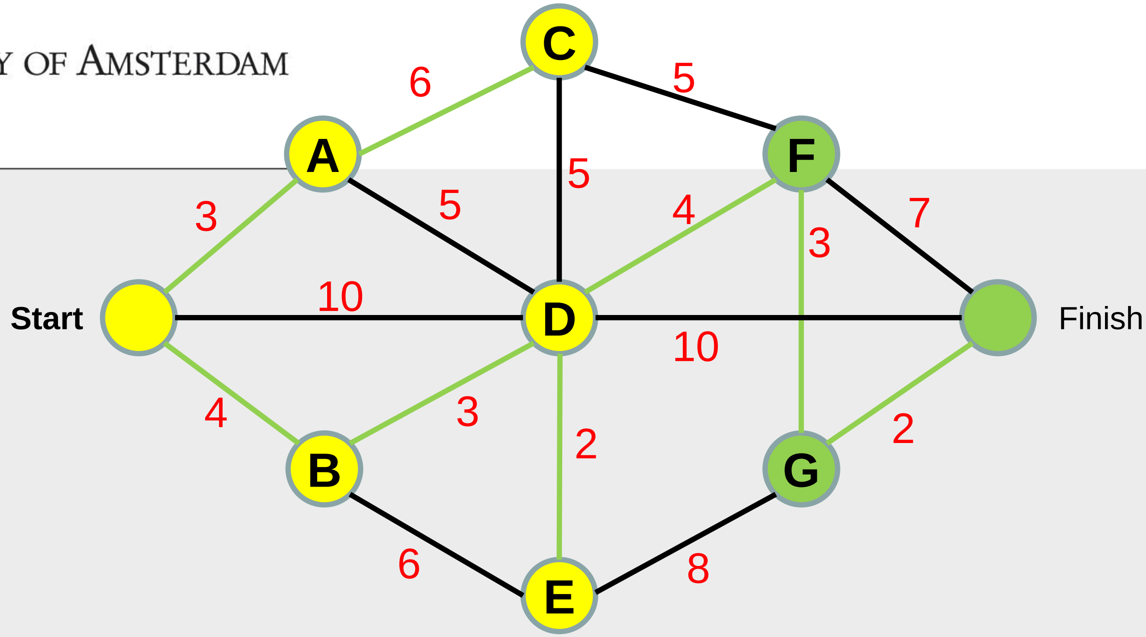


| A | B | C | D | E | F | G | Finish |
|---|---|---|---|---|---|---|---|
| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | **9, D** | 11, D | | 17, D |
| * | * | * | * | * | **11, D** | 17, E | 17, D |
| * | * | * | * | * | * | **14, F** | 17, D |
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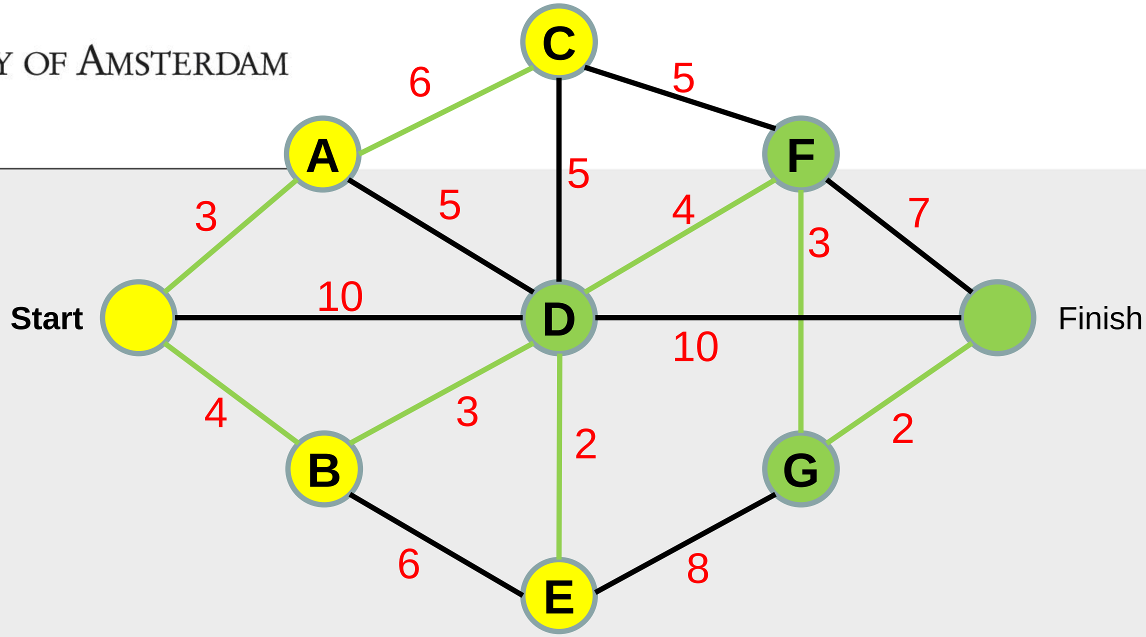


| A | B | C | D | E | F | G | Finish |
|---|---|---|---|---|---|---|---|
| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | **9, D** | 11, D | | 17, D |
| * | * | * | * | * | **11, D** | 17, E | 17, D |
| * | * | * | * | * | * | **14, F** | 17, D |
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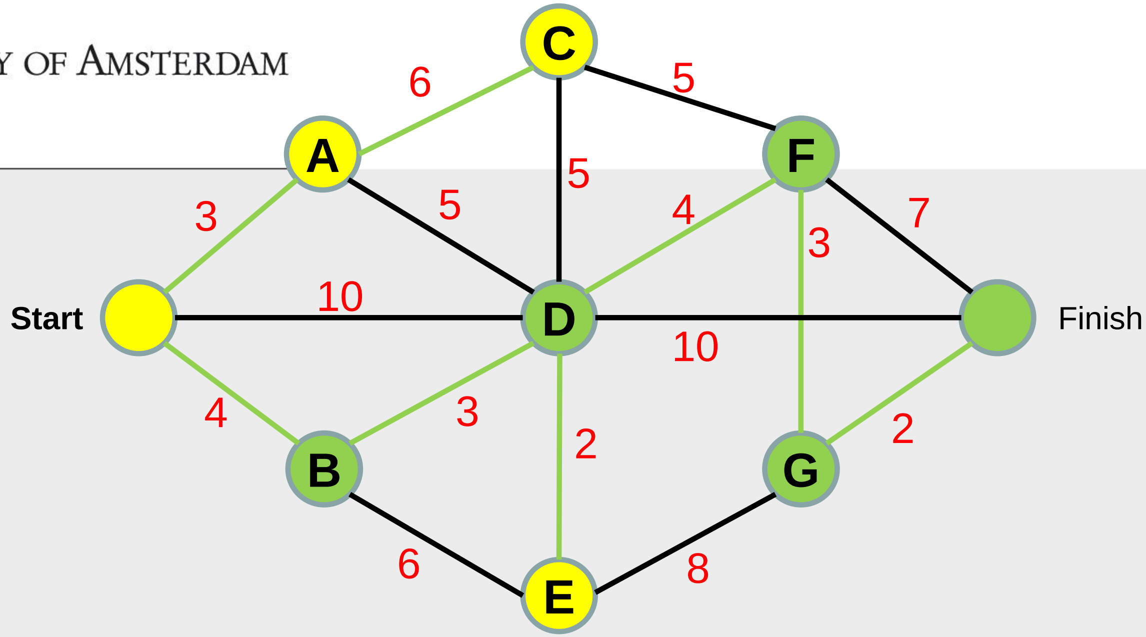
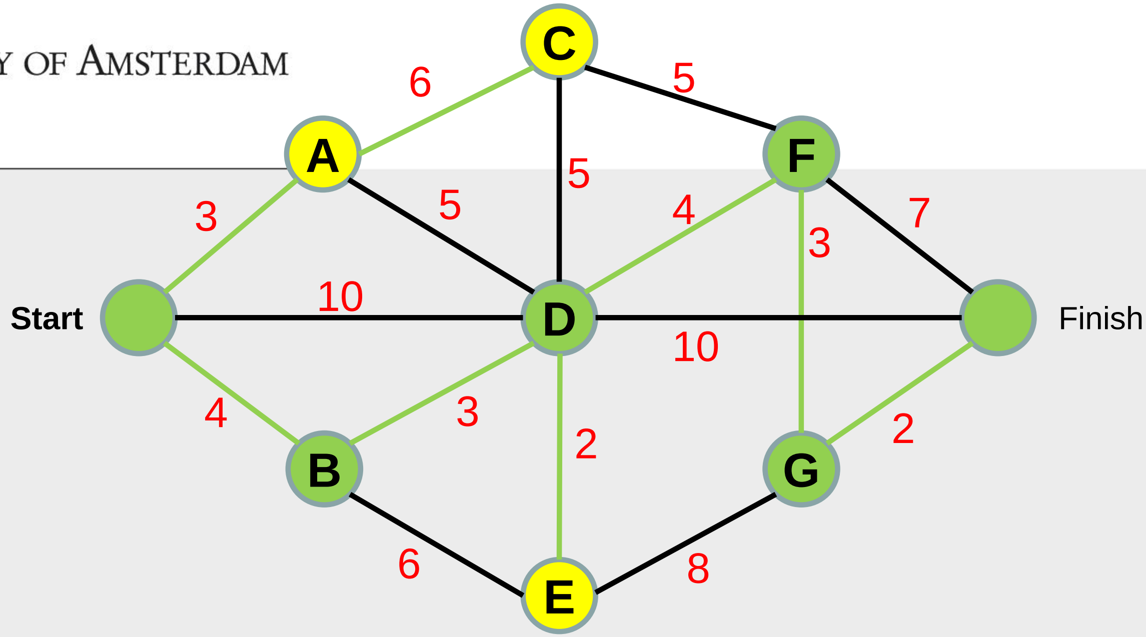
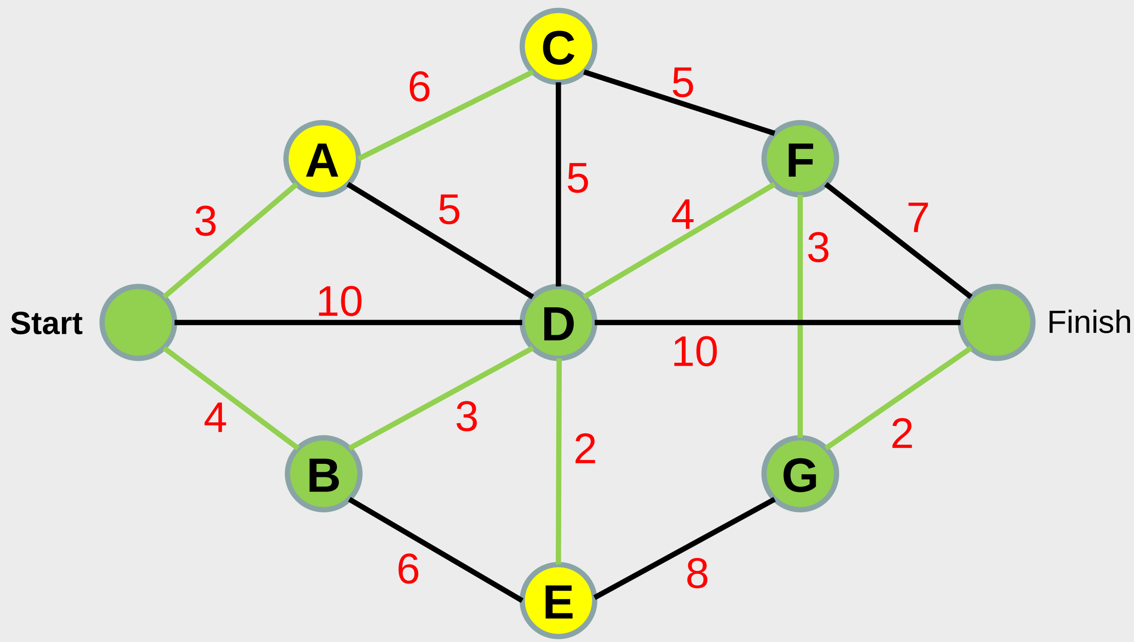



| A | B | C | D | E | F | G | Finish |
|---|---|---|---|---|---|---|---|
| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | **9, D** | 11, D | | 17, D |
| * | * | * | * | * | **11, D** | 17, E | 17, D |
| * | * | * | * | * | * | **14, F** | 17, D |
| * | * | * | * | * | * | * | **16, G** |



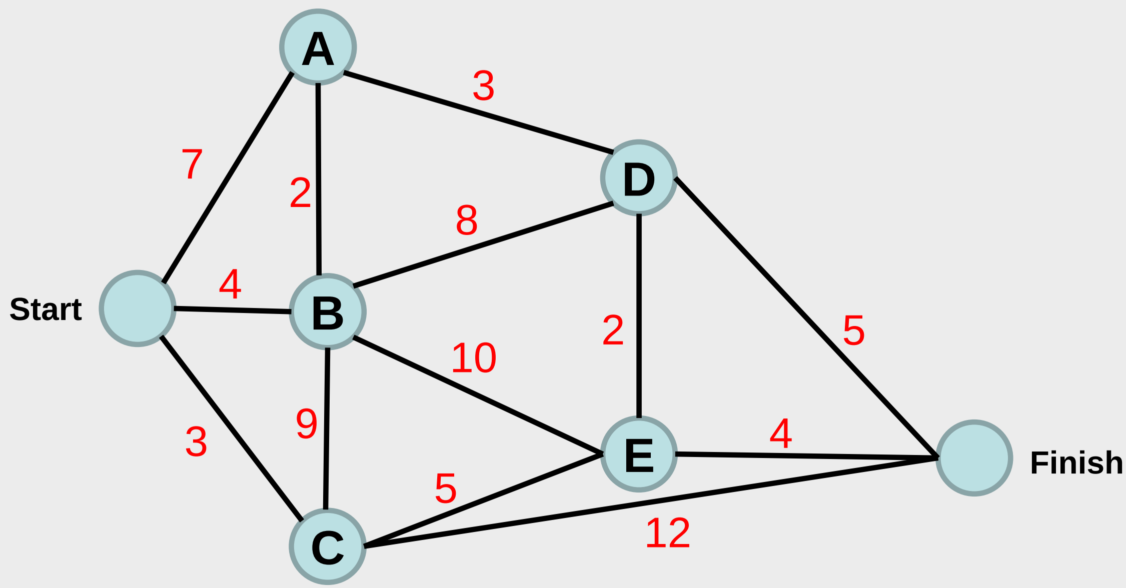

| A | B | C | D | E | F | G | Finish |
|---|---|---|---|---|---|---|---|
| **3, start** | 4, start | | 10, start | | | | |
| * | **4, start** | 9, A | 8, A | | | | |
| * | * | 9, A | **7, B** | 10,B | | | |
| * | * | **9, A** | * | 9, D | 11, D | | 17, D |
| * | * | * | * | **9, D** | 11, D | | 17, D |
| * | * | * | * | * | **11, D** | 17, E | 17, D |
| * | * | * | * | * | * | **14, F** | 17, D |
| * | * | * | * | * | * | * | **16, G** |

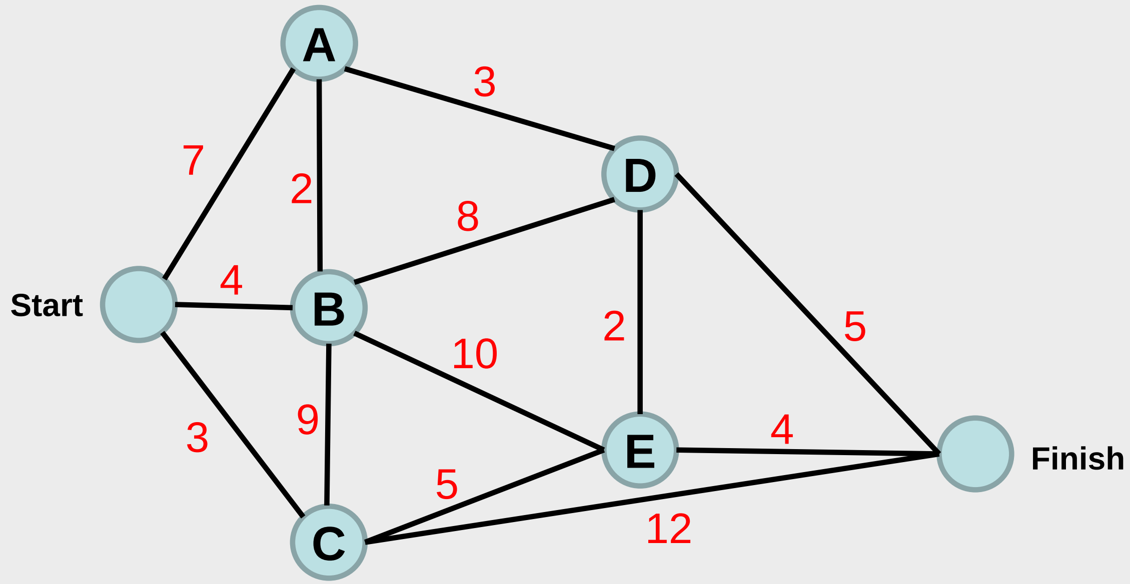



- The shortest route is given by: Start→B→D→F→G→Finish
- We made a mistake! We actually wanted to travel to node E!

# Exercise

- What is the shortest route from start to finish in the following network?






| A | B | C | D | E | Finish |
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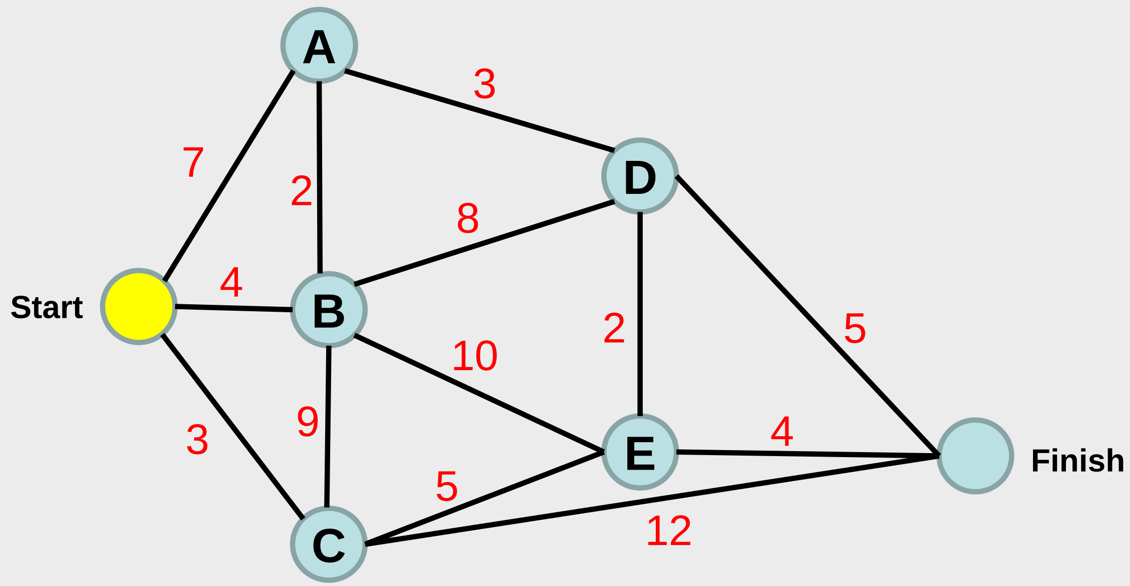


| A | B | C | D | E | Finish |
|---|---|---|---|---|---|
| 7, start | 4, start | 3, start | | | |
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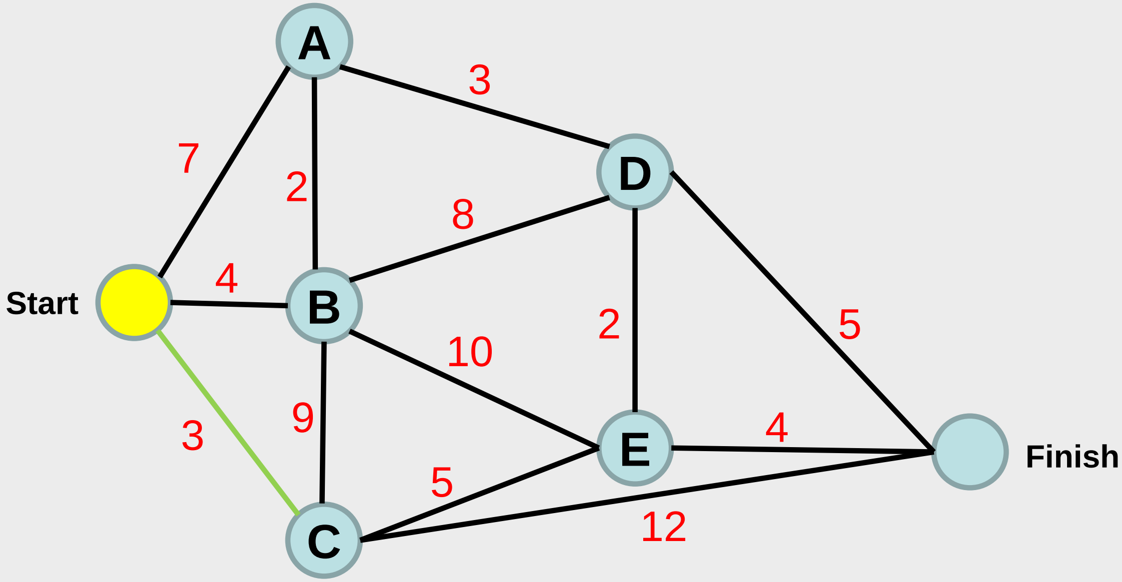


| A | B | C | D | E | Finish |
|---|---|---|---|---|---|
| 7, start | 4, start | **3, start** | | | |
| | | | | | |
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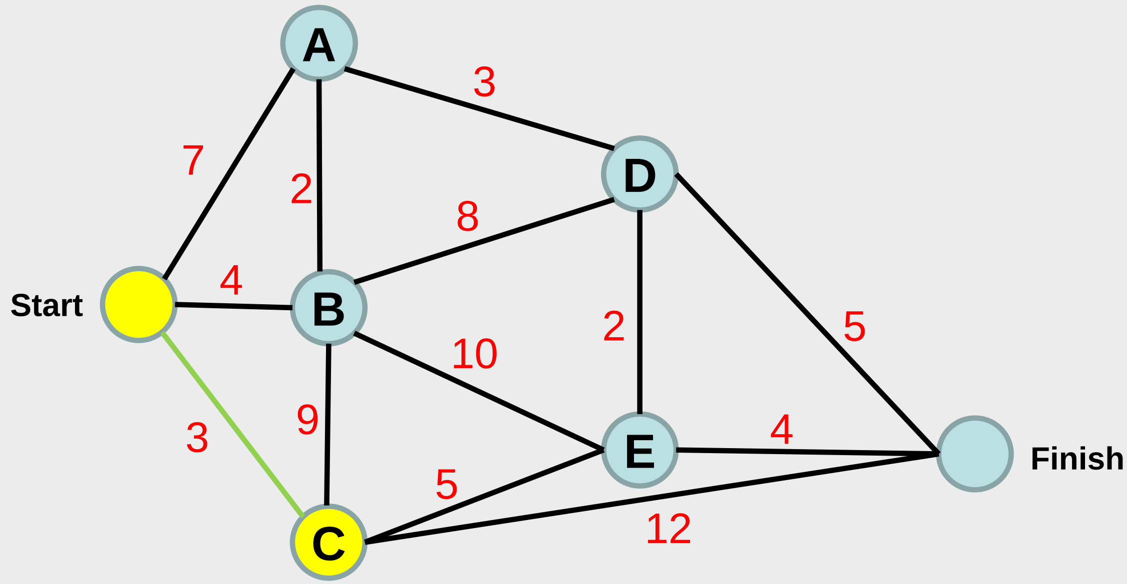
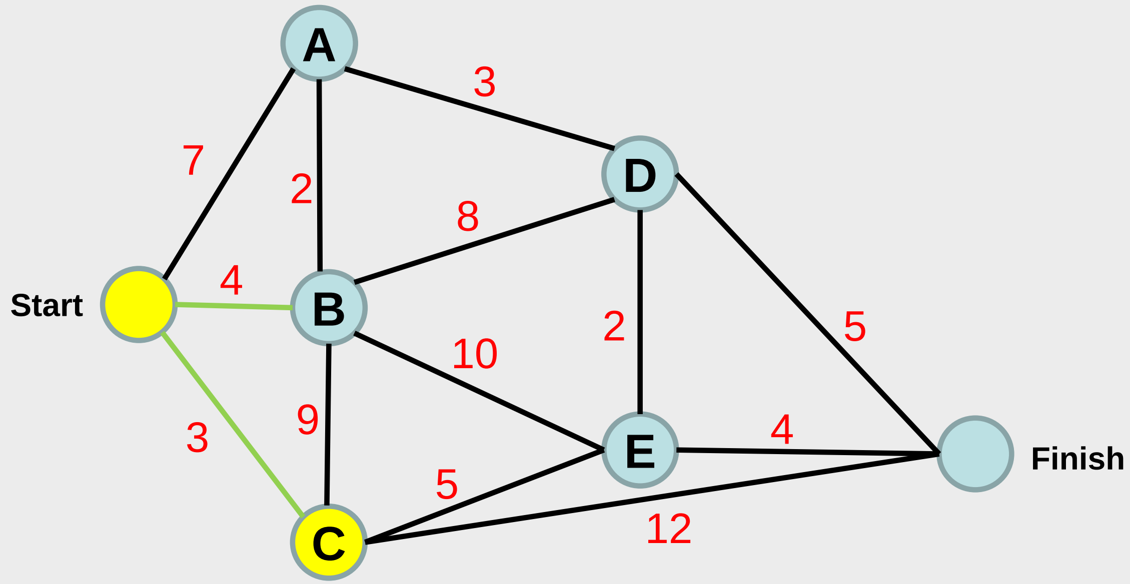
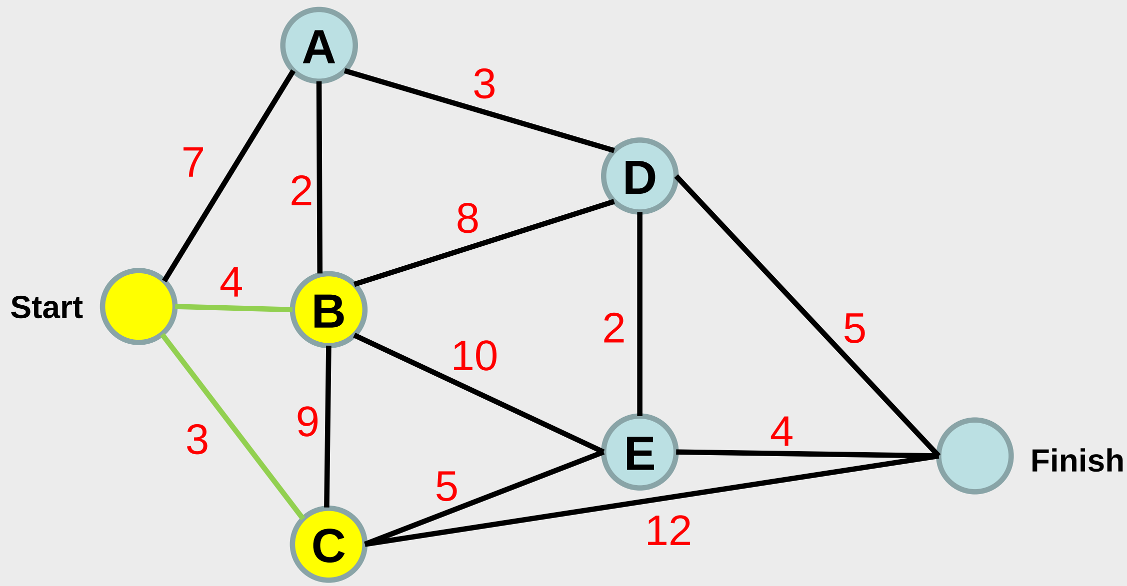


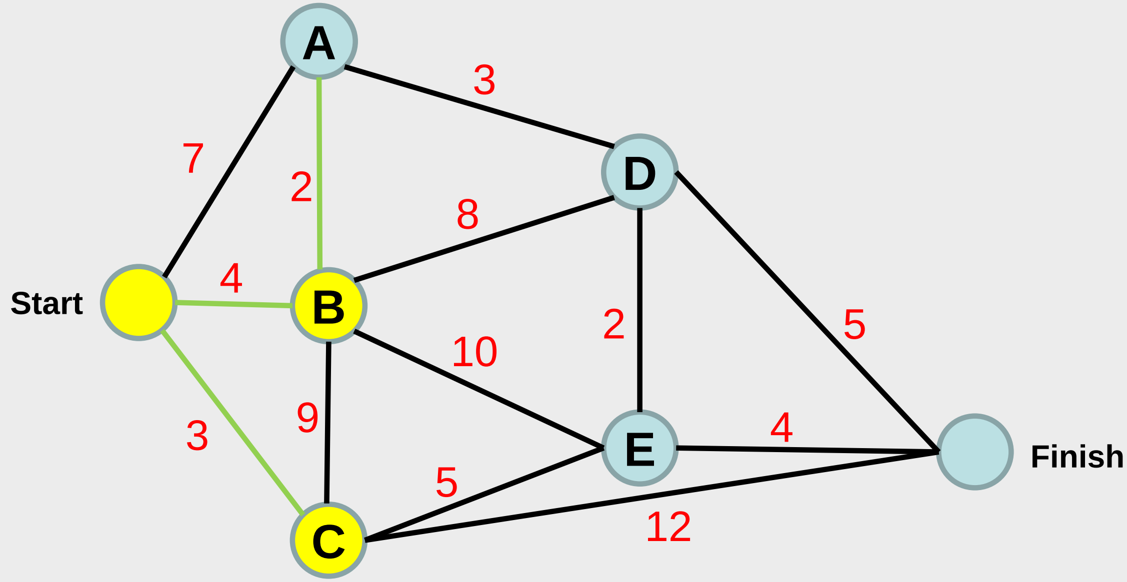

| A | B | C | D | E | Finish |
|---|---|---|---|---|---|
| 7, start | 4, start | **3, start** | | | |
| 7, start | 4, start | * | | 8, C | 15, C |
| | | * | | | |
| | | * | | | |
| | | * | | | |
| | | * | | | |



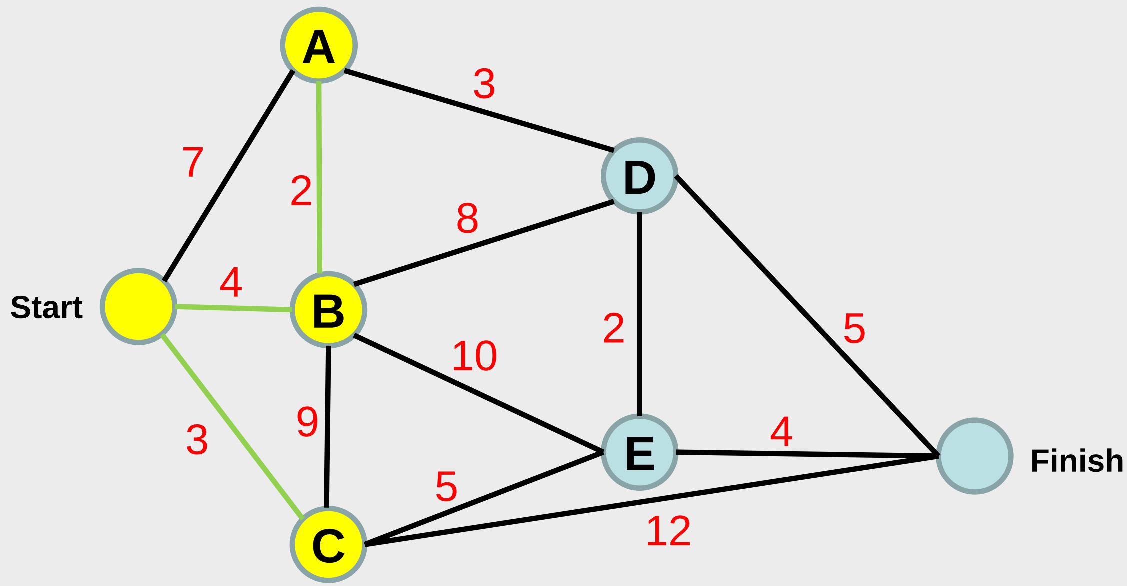

| A | B | C | D | E | Finish |
|---|---|---|---|---|---|
| 7, start | 4, start | **3, start** | | | |
| 7, start | **4, start** | * | | 8, C | 15, C |
| | | * | | | |
| | | * | | | |
| | | * | | | |
| | | * | | | |



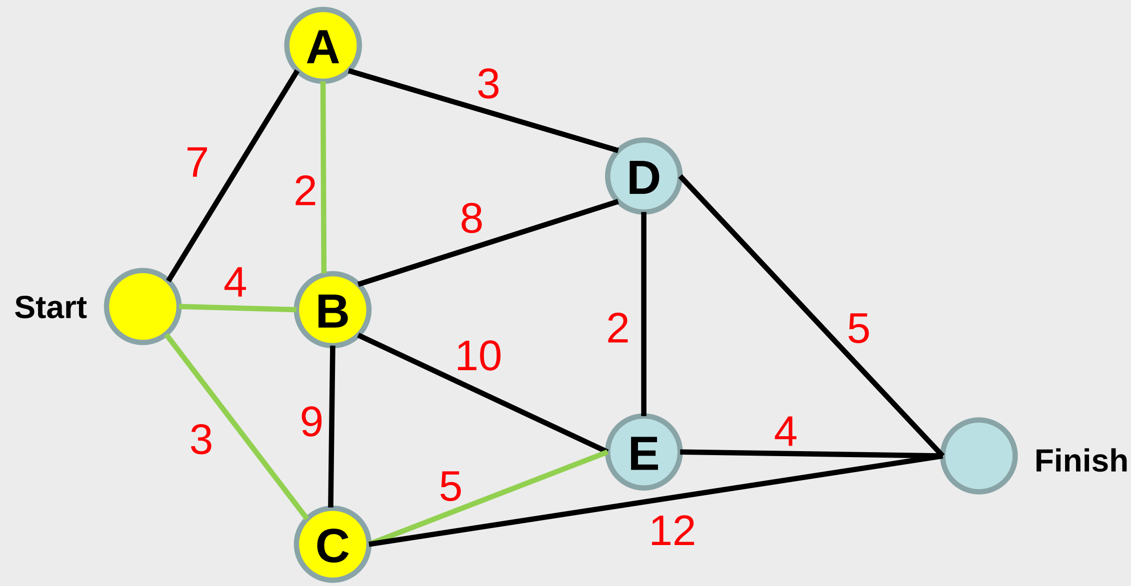

| A | B | C | D | E | Finish |
|---|---|---|---|---|---|
| 7, start | 4, start | **3, start** | | | |
| 7, start | **4, start** | * | | 8, C | 15, C |
| 6, B | * | * | 12, B | 8, C | 15, C |
| | * | * | | | |
| | * | * | | | |
| | * | * | | | |



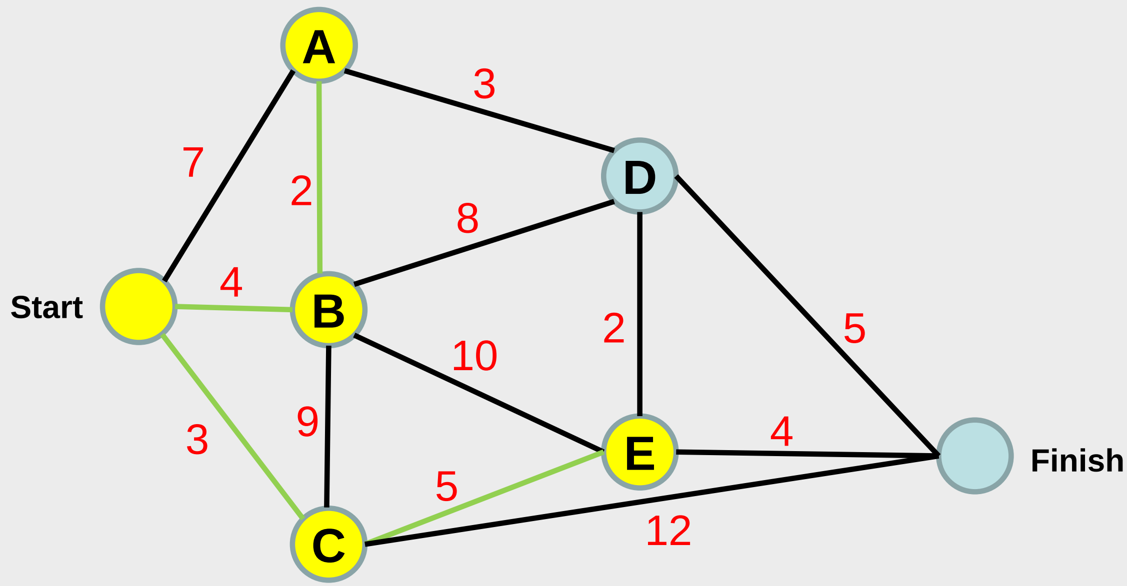

| A | B | C | D | E | Finish |
|---|---|---|---|---|---|
| 7, start | 4, start | **3, start** | | | |
| 7, start | **4, start** | * | | 8, C | 15, C |
| **6, B** | * | * | 12, B | 8, C | 15, C |
| | * | * | | | |
| | * | * | | | |
| | * | * | | | |



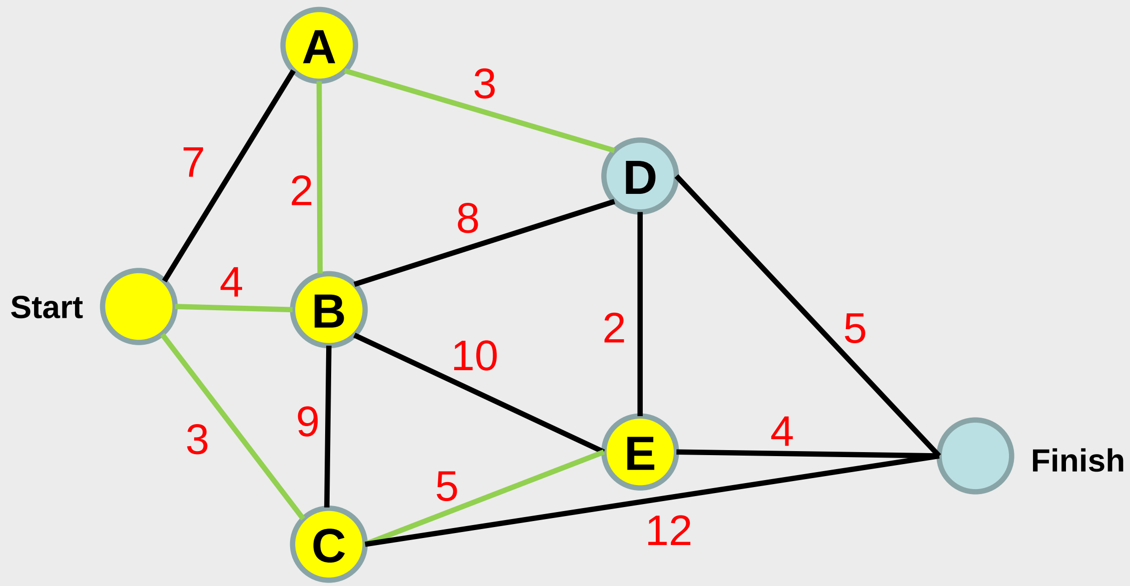

| A | B | C | D | E | Finish |
|---|---|---|---|---|---|
| 7, start | 4, start | **3, start** | | | |
| 7, start | **4, start** | * | | 8, C | 15, C |
| **6, B** | * | * | 12, B | 8, C | 15, C |
| * | * | * | 9, A | 8, C | 15, C |
| * | * | * | | | |
| * | * | * | | | |



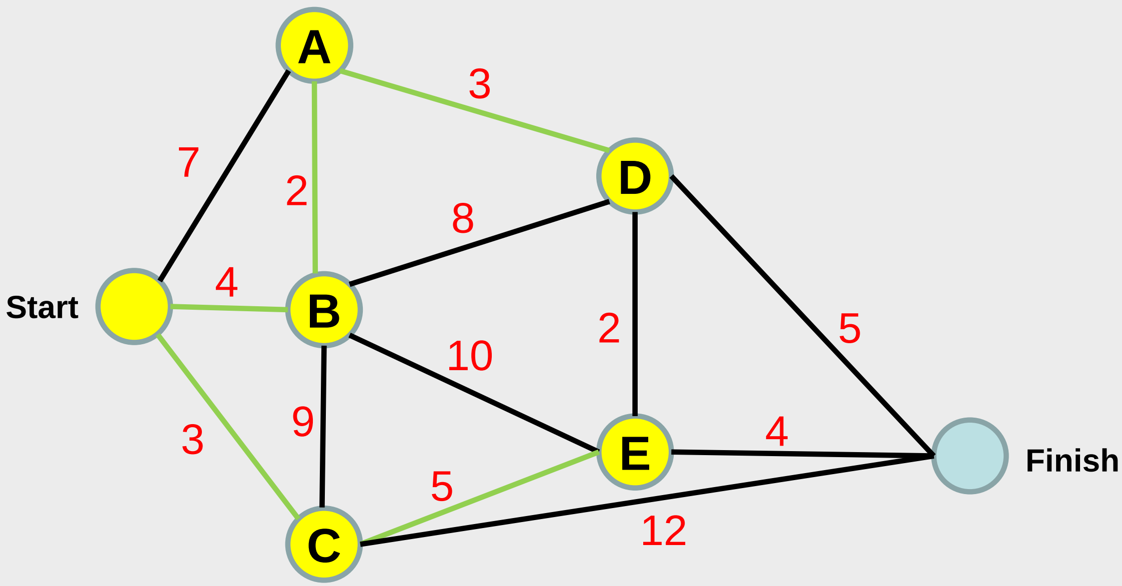

| A | B | C | D | E | Finish |
|---|---|---|---|---|---|
| 7, start | 4, start | **3, start** | | | |
| 7, start | **4, start** | * | | 8, C | 15, C |
| **6, B** | * | * | 12, B | 8, C | 15, C |
| * | * | * | 9, A | **8, C** | 15, C |
| * | * | * | | | |
| * | * | * | | | |



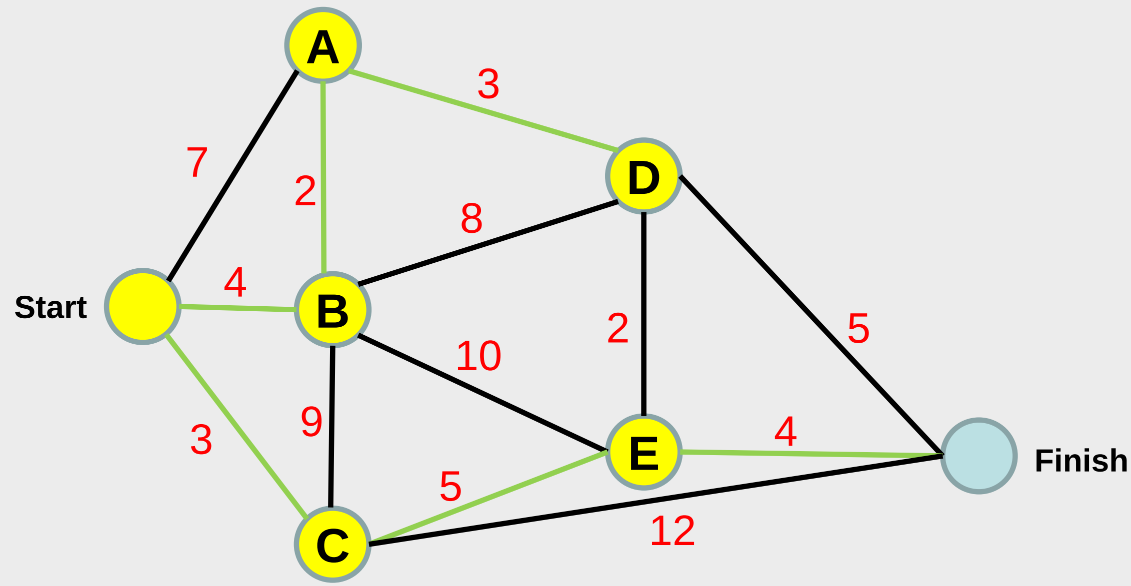

| A | B | C | D | E | Finish |
|---|---|---|---|---|---|
| 7, start | 4, start | **3, start** | | | |
| 7, start | **4, start** | * | | 8, C | 15, C |
| **6, B** | * | * | 12, B | 8, C | 15, C |
| * | * | * | 9, A | **8, C** | 15, C |
| * | * | * | 9, A | * | 12, E |
| * | * | * | | * | |



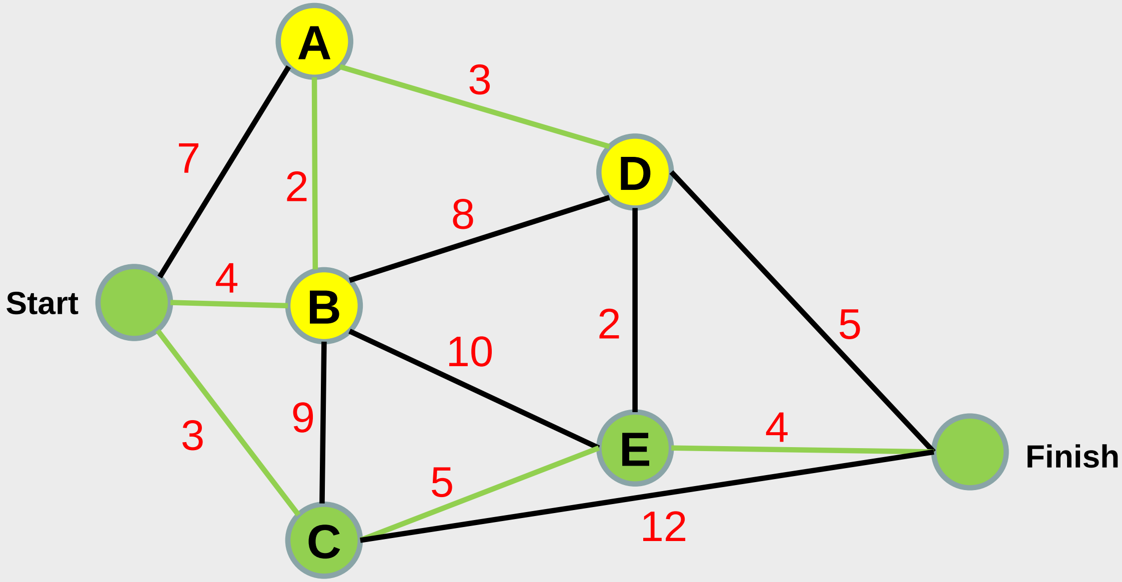

| A | B | C | D | E | Finish |
|---|---|---|---|---|---|
| 7, start | 4, start | **3, start** | | | |
| 7, start | **4, start** | * | | 8, C | 15, C |
| **6, B** | * | * | 12, B | 8, C | 15, C |
| * | * | * | 9, A | **8, C** | 15, C |
| * | * | * | **9, A** | * | 12, E |
| * | * | * | | * | |



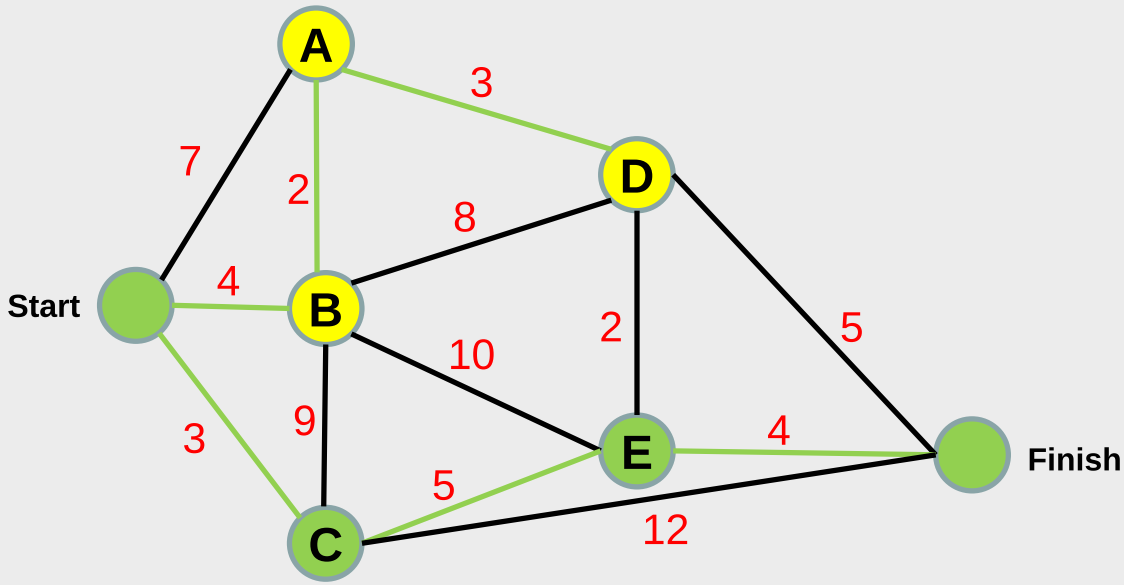

| A | B | C | D | E | Finish |
|---|---|---|---|---|---|
| 7, start | 4, start | **3, start** | | | |
| 7, start | **4, start** | * | | 8, C | 15, C |
| **6, B** | * | * | 12, B | 8, C | 15, C |
| * | * | * | 9, A | **8, C** | 15, C |
| * | * | * | **9, A** | * | 12, E |
| * | * | * | * | * | 12, E |




| A | B | C | D | E | Finish |
|---|---|---|---|---|---|
| 7, start | 4, start | **3, start** | | | |
| 7, start | **4, start** | * | | 8, C | 15, C |
| **6, B** | * | * | 12, B | 8, C | 15, C |
| * | * | * | 9, A | **8, C** | 15, C |
| * | * | * | **9, A** | * | 12, E |
| * | * | * | * | * | **12, E** |




| A | B | C | D | E | Finish |
| --- | --- | --- | --- | --- | --- |
| 7, start | 4, start | **3, start** | | | |
| 7, start | **4, start** | * | | 8, C | 15, C |
| **6, B** | * | * | 12, B | 8, C | 15, C |
| * | * | * | 9, A | **8, C** | 15, C |
| * | * | * | **9, A** | * | 12, E |
| * | * | * | * | * | **12, E** |




- The shortest route is given by: Start→C→E→Finish

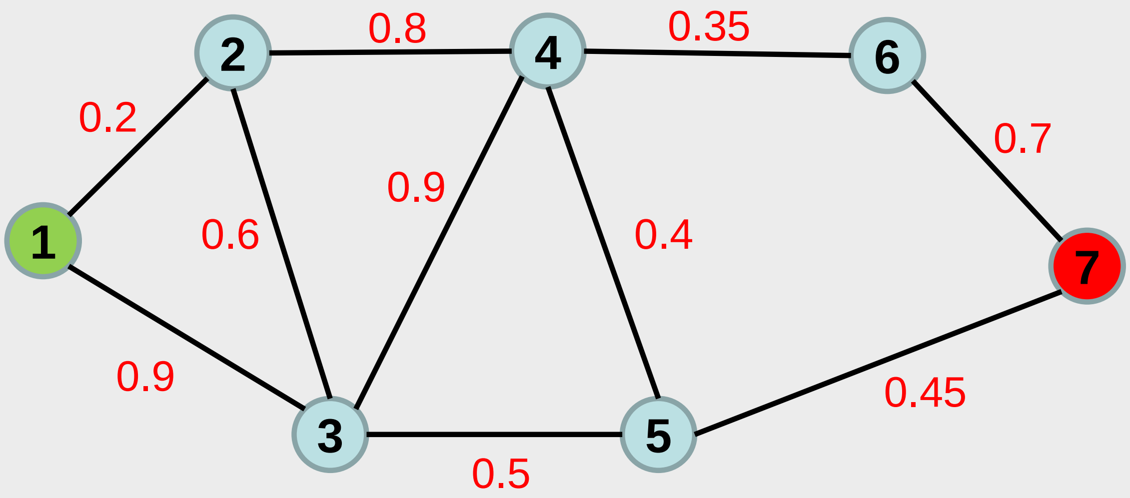
# Ten-minute break

# Introducing uncertainty

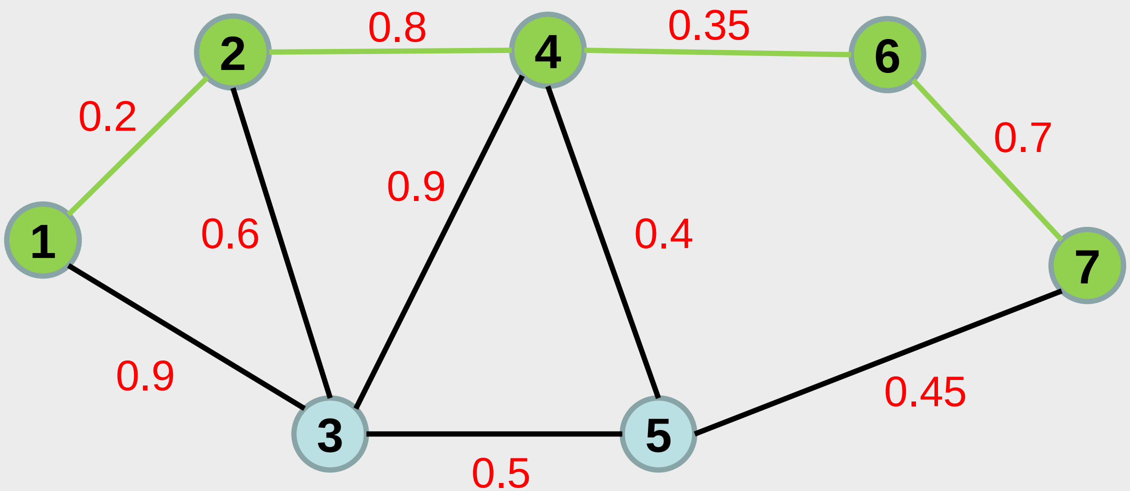
- So far, we have only been interested in finding the shortest route
- Perhaps there are routes other than the shortest route that are ‘better’
- The shortest route would not be a very desireable route if it is likely to suffer from traffic jams…

# Introducing uncertainty

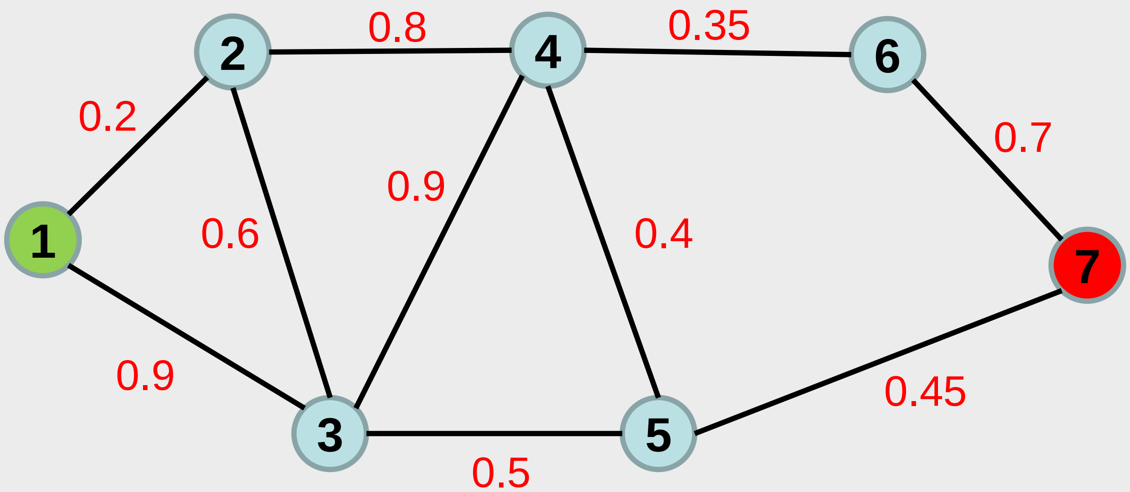
- Finding the route from 1 to 7 that has the lowest probability of getting stuck in a traffic jam




- The numbers denote the probability of not getting stuck in a traffic jam




- Consider the route $1 \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow 7$
- The probability of not getting stuck in a traffic jam is
  $$p_{17} = p_{12} \times p_{24} \times p_{46} \times p_{67}$$
- So, $p_{17} = 0.2 \times 0.8 \times 0.35 \times 0.7 \approx 0.04$
- Can we directly apply Dijkstra's algorithm to find the 'most-reliable' route?




- Note that we want to **maximise** a **product**

- We need to find a way to rewrite our problem such that is becomes a problem of mimimising a sum

# Applying Dijkstra in a non-standard setting

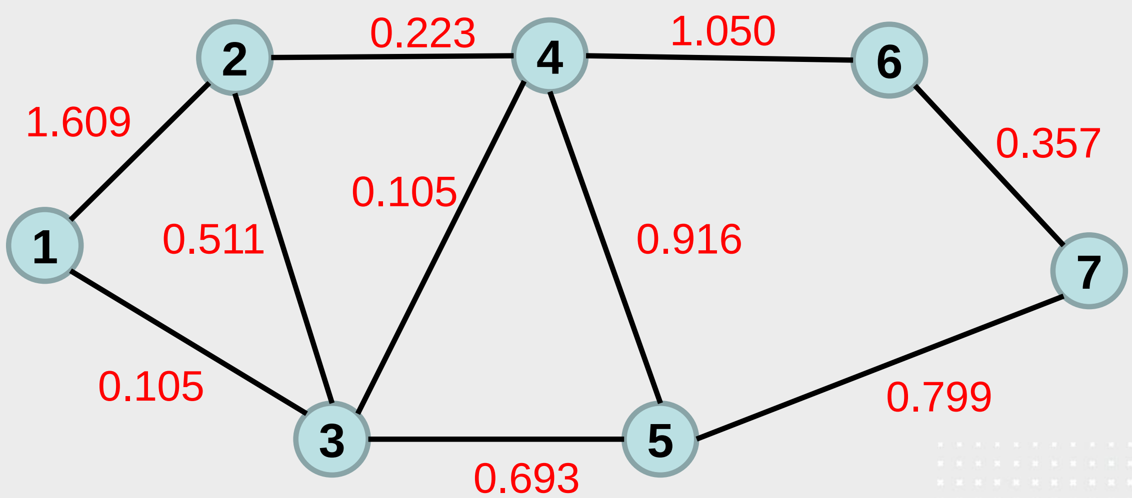
- Recall that $\log(xy) = \log(x) + \log(y)$
- Therefore,
  $$\log(p_{17}) = \log(p_{12}) + \log(p_{24}) + \log(p_{46}) + \log(p_{67})$$
- But we are interested in the route that maximises $p_{17}$, not the route that maximises the log of $p_{17}$
- Note that the logarithm is a strictly increasing function
- Why is this important?

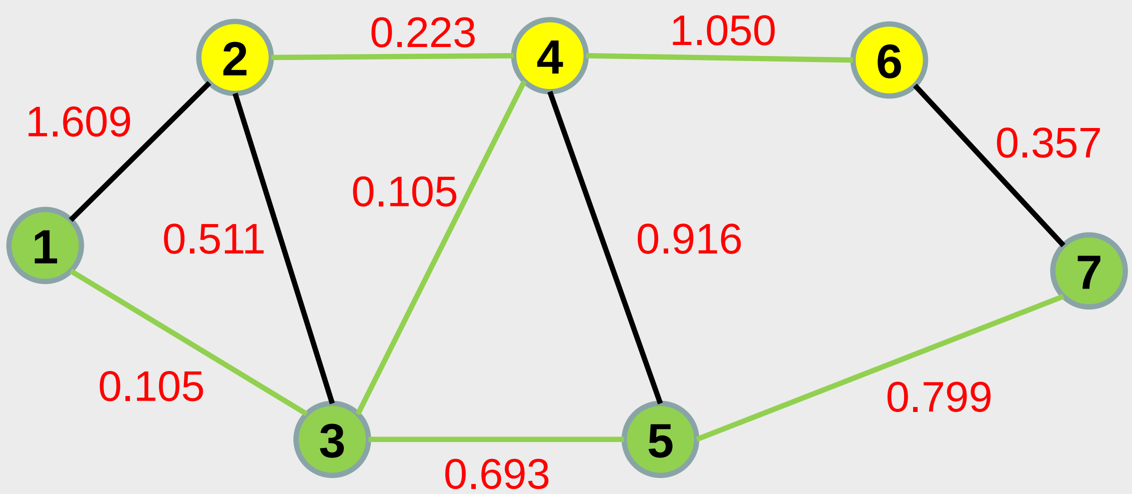
# Applying Dijkstra in a non-standard setting

- Recall that Dijkstra's algorithm minimises a sum, but we want to maximise a sum
- We can make use of the following trick: The route that maximises $\log(p_{17})$ will minimise $-\log(p_{17})$
- Conclusion: We can use Dijkstra to find the route that minimises $-\log(p_{17})$ and this is the route we are interested in finding

# Exercise

- Find the route that minimises the probability of getting stuck in a traffic jam
- We have already applied the –log transformation for you:

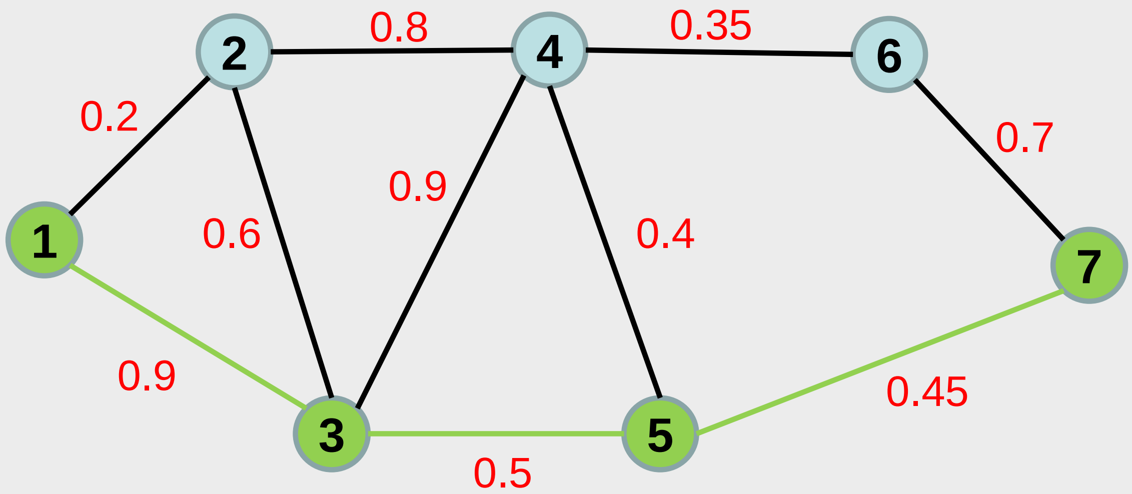





| nodes | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| Step 1 | $(1.609, 1)$ | $(0.105, 1)^*$ | No edge | No edge | No edge | No edge |
| Step 2 | $(0.616, 3)$ | - | $(0.210, 3)^*$ | $(0.798, 3)$ | No edge | No edge |
| Step 3 | $(0.433, 4)^*$ | - | - | $(0.798, 3)$ | $(1.260, 4)$ | No edge |
| Step 4 | - | - | - | $(0.798, 3)^*$ | $(1.260, 4)$ | No edge |
| Step 5 | - | - | - | - | $(1.260, 4)^*$ | $(1.597, 5)$ |
| Step 6 | - | - | - | - | - | $(1.597, 5)^*$ |

*Table 2.2.1. Dijkstra's shortest route algorithm for the network in Figure 2.2.2.*

- The path that minimises $-\log(p_{17})$, maximises $p_{17}$




- We conclude that we found a route such that the probability of not getting stuck in a traffic jam is

$$p_{17} = 0.9 \times 0.5 \times 0.45 \approx 0.20$$

# Refinements

- Up to now, we have considered two extremes
  - The shortest route, not taking into account the reliability of the route
  - The most reliable route, not taking into account the length of the route
- We would like to find a route that is both quick and reliable…

# Travel times as random variable

- Instead of distances, we are now going to work with travel times
- Travel times can be uncertain for many reasons
- It makes sense to model the travel times as random variables
- We will assume the travel times are normally distributed

# The density of a normally distributed r.v.

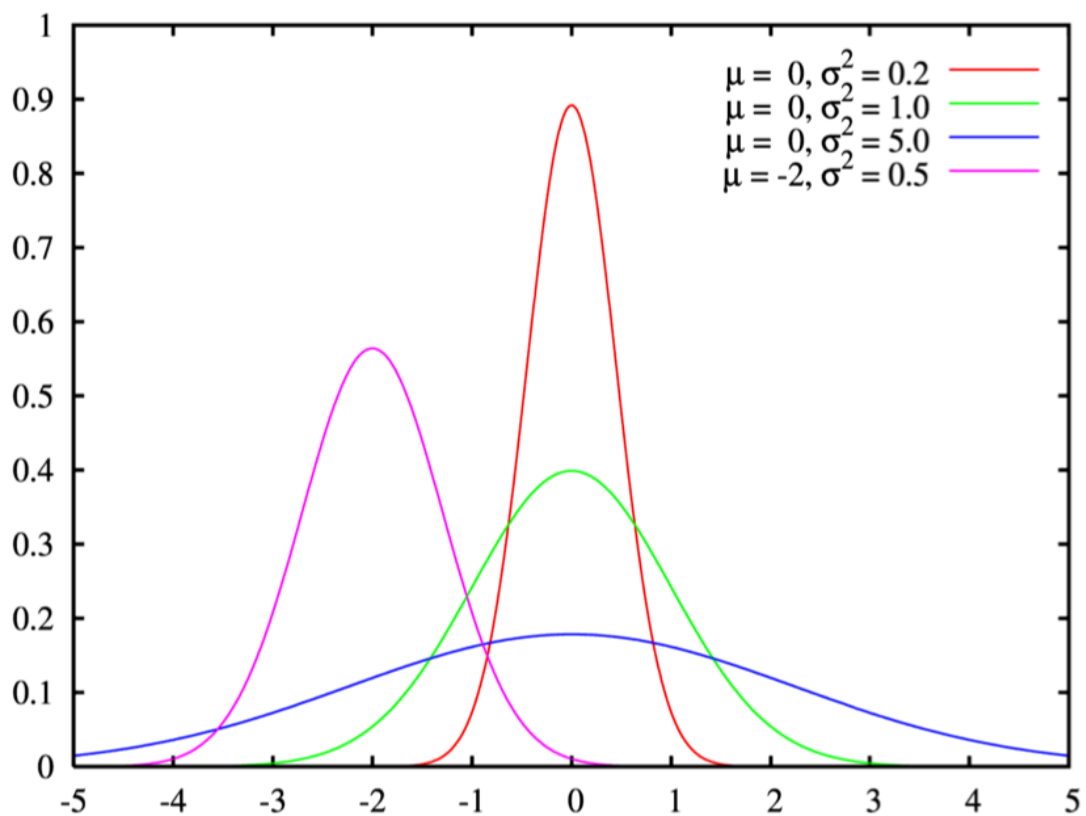
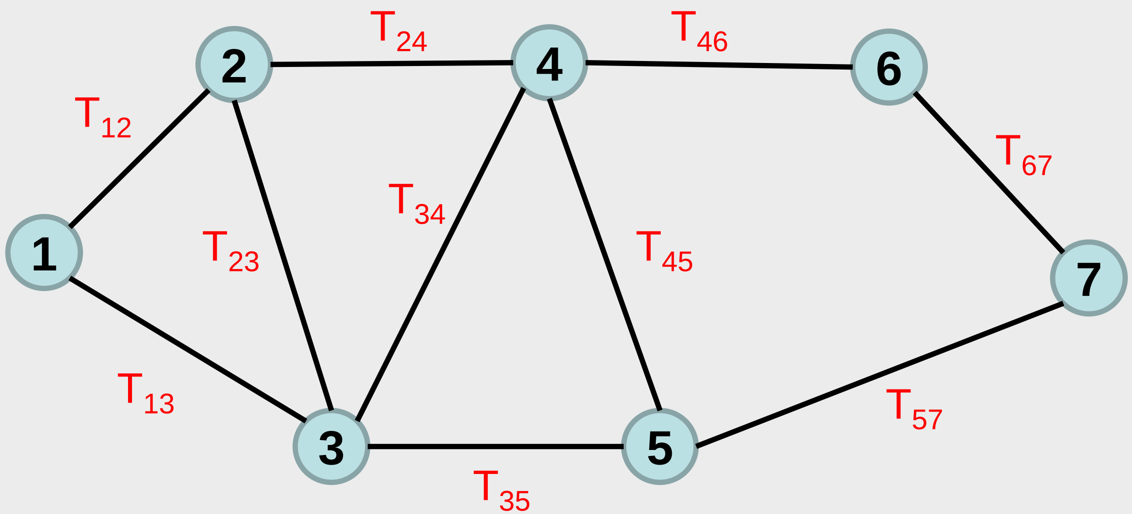



*Figure 1.2.1. The density function of a normally distributed random variable with parameters $\mu$ and $\sigma^2$.*

# A road travel network with stochastic travel times




- The travel times $T_{ij}$ have a mean and a variance
- Suppose that $T_{13}$ has mean 15 and variance 4
- Suppose that $T_{12}$ has mean 12 and variance 16

# Quiz time!

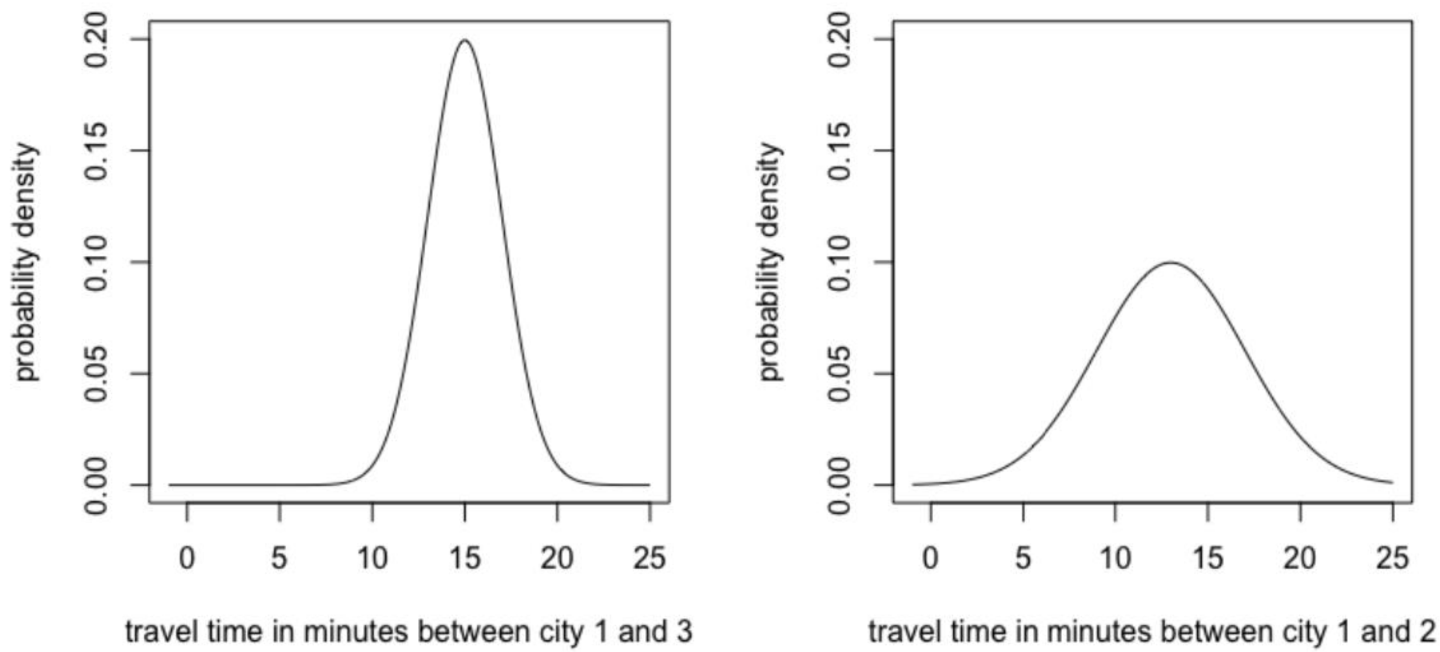



*Figure 2.2.4. The density of* $T_{13} \sim \mathcal{N}(15, 4)$ *(left) and* $T_{12} \sim \mathcal{N}(12, 16)$ *(right).*

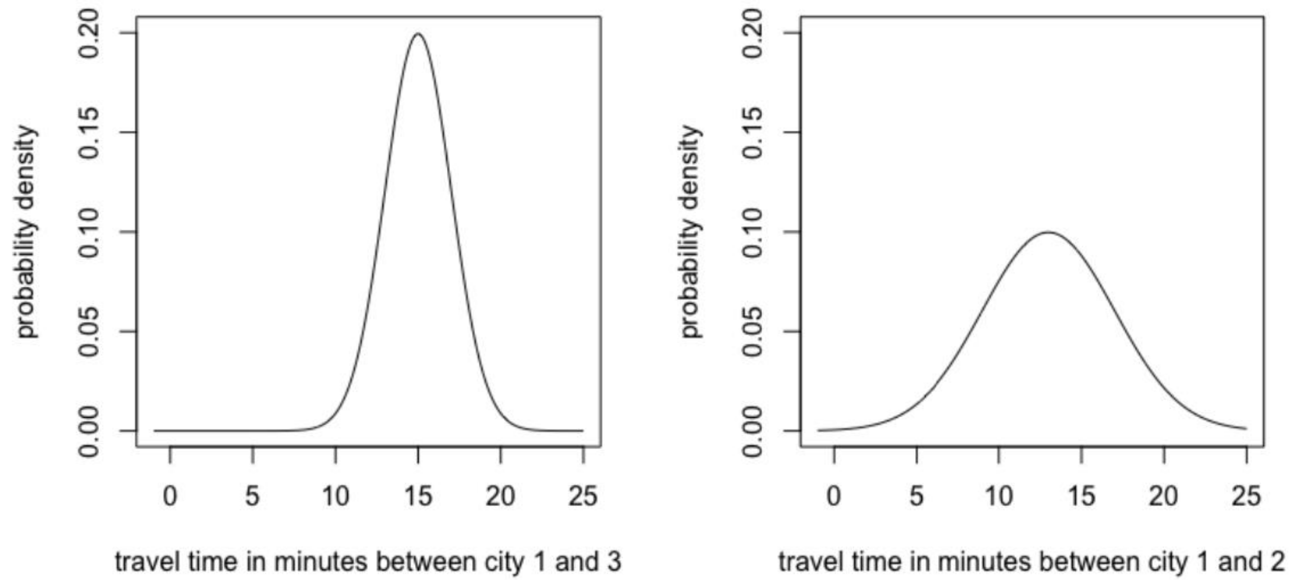
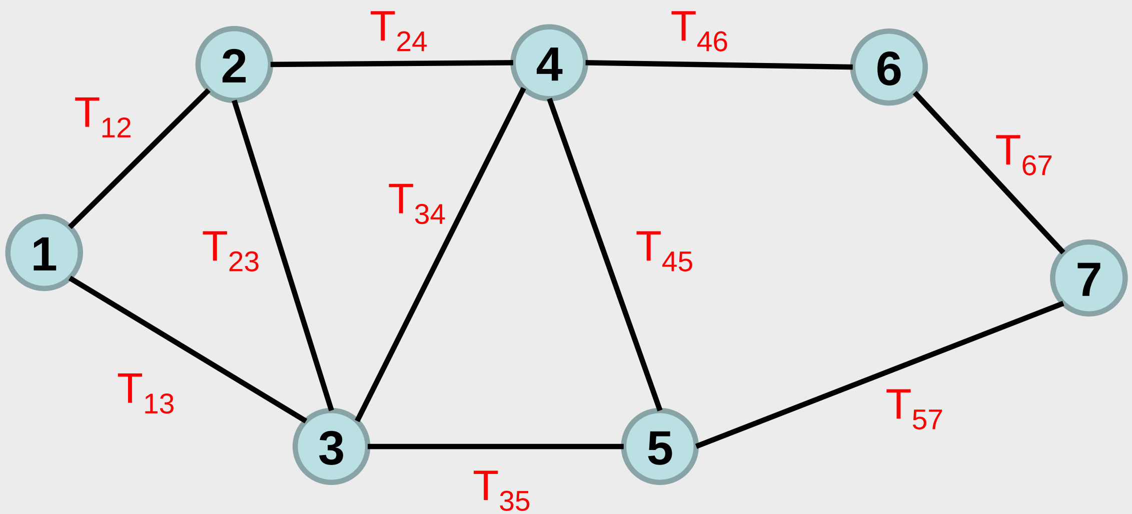



*Figure 2.2.4. The density of* $T_{13} \sim \mathcal{N}(15, 4)$ *(left) and* $T_{12} \sim \mathcal{N}(12, 16)$ *(right).*

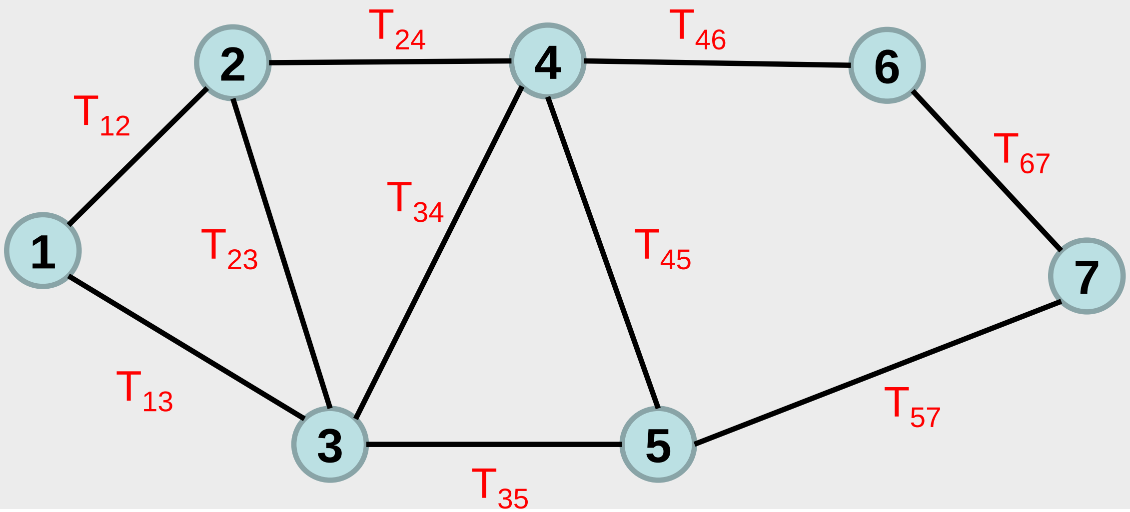
- The travel time to the left has a higher mean, so it is expected to be faster
- The travel time to the right has a higher variance, so it is less reliable

# Ten-minute break

# A road travel network with stochastic travel times




- The travel times $T_{ij}$ have a mean and a variance

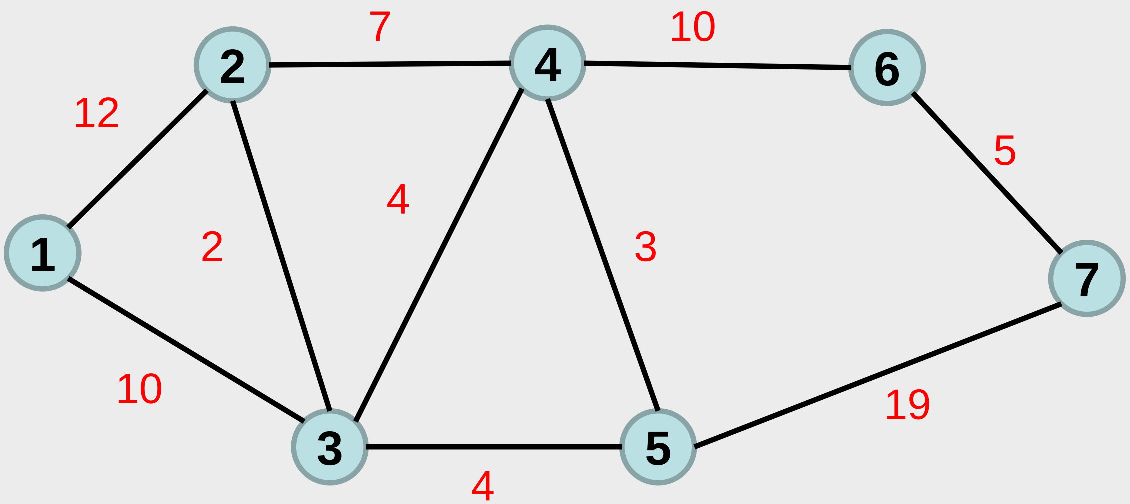



| | $T_{12}$ | $T_{13}$ | $T_{23}$ | $T_{24}$ | $T_{34}$ | $T_{35}$ | $T_{45}$ | $T_{46}$ | $T_{57}$ | $T_{67}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mu$ | 12 | 10 | 2 | 7 | 4 | 4 | 3 | 10 | 19 | 5 |
| $\sigma^2$ | 1 | 9 | 1 | 9 | 4 | 1 | 1 | 16 | 1 | 4 |

*Table 3.3.1. $T_{ij}$ denotes the travel time between city $i$ and $j$ in minutes and is normally distributed with parameters $\mu$ and $\sigma^2$.*

- We can easily find the route with the lowest expected travel time by applying Dijkstra

- Simply replace each random variable $T_{ij}$ by its mean




- Animation!

- The path with shortest expected travel time is
  1→3→4→6→7

# Arriving in time

- We have seen that we are able to determine the route with the lowest expected travel time
- It may make more sense to find the route that maximises the probability of arriving at your destination in time
- This could be a different route!

# Computing probabilities

- We have argued that $T_{12}$ (which has mean 12 and variance 16) is expected to be a quicker road piece to traverse compared to $T_{13}$, but it is also less reliable
- Lets compute $P(T_{12} < 20)$:
- $P(T_{12} < 20) = P(\frac{T_{12}-12}{\sqrt{16}} < \frac{20-12}{\sqrt{16}})$
- Since $\frac{20-12}{\sqrt{16}} = 2.00$, the table on page 50 of your booklet tells us that $P(T_{12} < 20) \approx 0.9772$

# Quiz

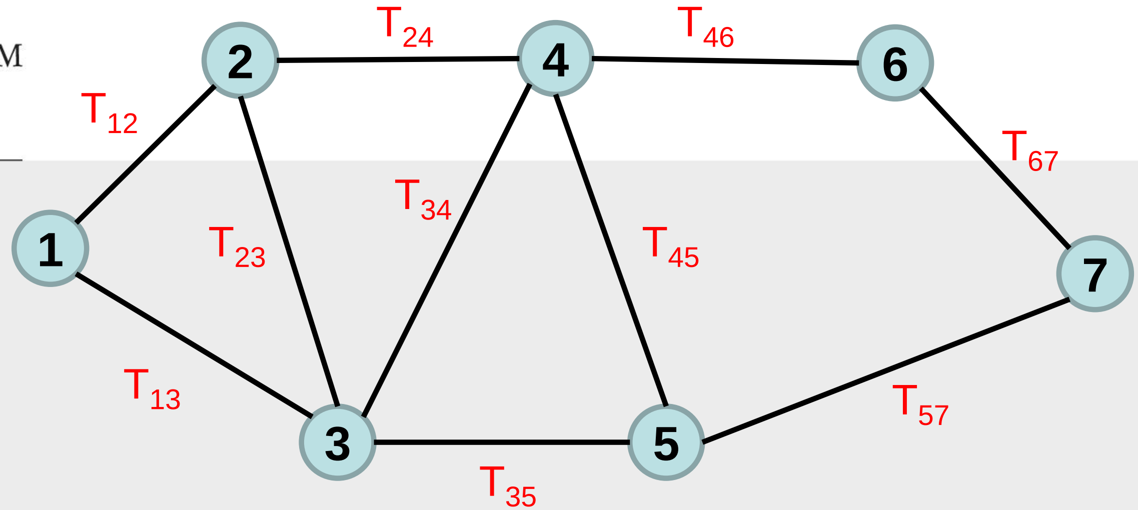
- Compute $P(T_{13} < 20)$
- Recall that $T_{13}$ has mean 15 and variance 4. Use the table on page 50 in your booklet

# Random variables compared

- We have found that
  - $P(T_{12} < 20) \approx 0.9772$
  - $P(T_{13} < 20) \approx 0.9938$
- It is remarkable that the road piece connecting 1 and 2 is less likely to be traversed within 20 minutes, while this road piece has a lower expected travel time!

# From road pieces to routes

- This idea can be pushed further
- Instead of road pieces, we may also find the route that maximises the probability of arriving on time
- This route may be different than the route that minimises the expected travel time we have determined before




- Recall that the route with the lowest expected travel time is 1→3→4→6→7. What is the probability you arrive at your destination within 40 minutes? Also compute this probability for the route 1→3→5→7. Use the hint.

| | $T_{12}$ | $T_{13}$ | $T_{23}$ | $T_{24}$ | $T_{34}$ | $T_{35}$ | $T_{45}$ | $T_{46}$ | $T_{57}$ | $T_{67}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mu$ | 12 | 10 | 2 | 7 | 4 | 4 | 3 | 10 | 19 | 5 |
| $\sigma^2$ | 1 | 9 | 1 | 9 | 4 | 1 | 1 | 16 | 1 | 4 |

*Table 3.3.1. $T_{ij}$ denotes the travel time between city $i$ and $j$ in minutes and is normally distributed with parameters $\mu$ and $\sigma^2$.*

**Hint**

Recall that the travel times are independent across the different roads. You can use that if $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ are independent, it holds that $X + Y \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

- Note that the route 1→3→4→6→7 has mean 29 and variance 33
- This gives $P(T_{17} < 40) \approx 0.9719$
- The route 1→3→5→7 has mean 33 and variance 11
- This gives $P(T_{17} < 40) \approx 0.9826$
- The latter route has a higher probability of arriving in time!
- We can also use Dijkstra’s algorithm to find fast and reliable routes!

# Concluding remarks

- Dijkstra's algorithm is a very useful tool to find paths that minimise a certain quantity on a network
- Modelling travel times as random variables allows us to find quick routes, while also taking the reliability of the routes into account
- Routes with the lowest expected travel times, are not necessarily optimal!

# Thank you!