# Strategic decision making in an $M/M/1$ queue

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# Queues

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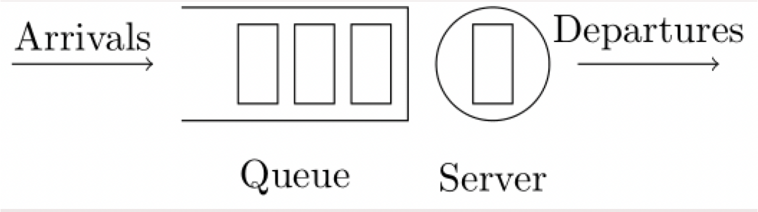
# Queueing theory

- Should I join the line or not?
- How can we reduce the waiting times?
- How much shall we charge for the service based on our service rate?

# Queueing theory

- (Customer perspective) - Should I join the line or not?
- (Social planner perspective) - How can we reduce the waiting times?
- (System manager perspective) - How much shall we charge for the service based on our service rate?

# $M/M/1$ queue



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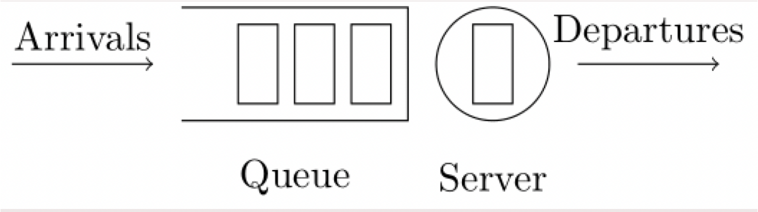


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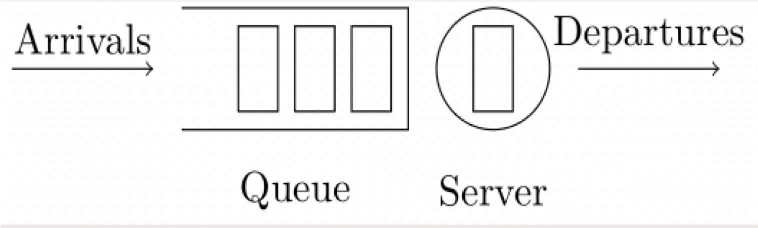


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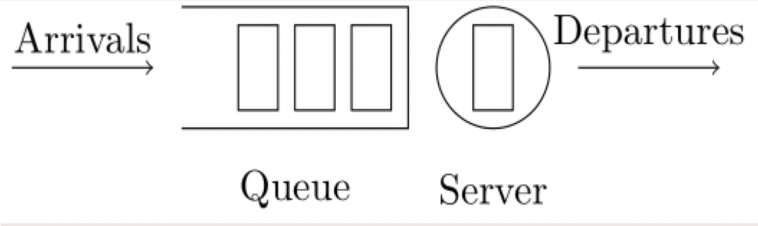


# $M/M/1$ queue


- $L$: total customers, which is a random variable
- $\lambda$: arrival rate
- $\mu$: departure rate

$$p_i = \mathbb{P}(L = i)$$

$$\lambda\, p_0 = \mu\, p_1$$

$$\lambda\, p_1 = \mu\, p_2$$

$$\lambda\, p_2 = \mu\, p_3$$

$$\dots$$

$$\lambda\, p_{i-1} = \mu\, p_i$$

# $M/M/1$ queue


- $L$: total customers
- $\lambda$: arrival rate
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$$\lambda\, p_0 = \mu\, p_1 \Rightarrow p_1 = \frac{\lambda}{\mu} p_0$$

$$\lambda\, p_1 = \mu\, p_2 \Rightarrow p_2 = \frac{\lambda}{\mu} p_1 = \left(\frac{\lambda}{\mu}\right)^2 p_0$$

$$\lambda\, p_2 = \mu\, p_3 \Rightarrow p_3 = \frac{\lambda}{\mu} p_2 = \left(\frac{\lambda}{\mu}\right)^3 p_0$$

$$\ldots \qquad \lambda\, p_{i-1} = \mu\, p_i \Rightarrow p_i = \frac{\lambda}{\mu} p_{i-1} = \left(\frac{\lambda}{\mu}\right)^i p_0$$

# $M/M/1$ queue

**Question**: We introduce $\rho = \lambda/\mu$ to make the calculations easier. In the $M|M|1$ queue,

- find the probability $p_i$, in terms of $\rho$,

# $M/M/1$ queue

- The sum $\sum_{i=0}^{\infty} p_i = 1$, thus

$$p_0 \sum_{i=0}^{\infty} = p_0 \frac{1}{1-\rho} = 1\,.$$

It follows that

$$p_0 = 1 - \rho, \qquad p_i = \rho^i \left(1 - \rho_0\right).$$

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2. $L$ can take values from $\{0, 1, \ldots\}$,
3. The probability that $L = i$ is from the previous question $p_i = (1-\rho)\rho^i$,
4. From the definition of expected value, $\mathbb{E}[L] = \sum_{i=0}^{\infty} ip_i$.

# $M/M/1$ queue

$$
\begin{aligned}
\mathbb{E}[L] &= \sum_{i=0}^{\infty} l\, p_i = \sum_{i=0}^{\infty} i(1-\rho)\rho^i \\
&= \rho(1-\rho) \sum_{i=1}^{\infty} i \rho^{i-1} \\
&= \rho(1-\rho) \left( \sum_{i=0}^{\infty} \rho^i \right)' \\
&= \rho(1-\rho) \left( \frac{1}{1-\rho} \right)' \\
&= \frac{\rho}{1-\rho}.
\end{aligned}
$$

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- The cost to a customer for staying in a queue (i.e. for queueing) is $C$ monetary units in unit time.
- Each arriving customer weighs the net gains associated with joining or not joining.

# Strategic customers - net gain

**Q**: What is the net gain, if a customer chooses to join, when she observes $i$ people in the system?

1. the service time is exponentially distributed with rate $\mu$.
2. the expected value of the service time is $\frac{1}{\mu}$.
3. the cost is $C$ monetary units in unit time, thus the expected cost of waiting for one customer finishing her service is $\frac{C}{\mu}$.
4. the total expected waiting cost is $C\,\frac{i}{\mu}$.

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The net gain is thus

$$G_i = R - C\frac{i}{\mu}\,.$$

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An arriving customer will choose to join if $L \leq n_s$.

Note that $n_s$ must be an integer, thus

$$n_s = \left\lfloor \frac{R\mu}{C} \right\rfloor,$$

which denotes the largest integer not exceeding $R\mu/C$. Note that $n_s$ depends on $\mu, R, C$, but not on the arrival rate $\lambda$.

# $M/M/1/k$ queue

The $M/M/1/k$ queue is a finite-capacity queueing system.

- The system has a finite capacity of $k$ (including the customer being served).
- Only one job receives service at a time.
- Whenever a job arrives and the system is full, it will be blocked and it will leave forever.

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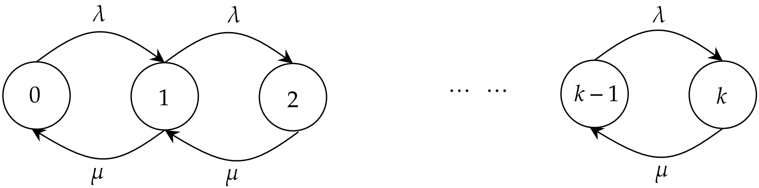

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$$\sum_{i=0}^{k} p_i = \sum_{i=0}^{k} p_0\,\rho^i = 1$$

Thus

$$p_0 = \frac{1-\rho}{1-\rho^{k+1}} \qquad p_i = \frac{(1-\rho)\rho^i}{1-\rho^{k+1}}$$

- the probability that an arriving customer is blocked.

$$p_b := p_k = \frac{(1-\rho)\rho^k}{1-\rho^{k+1}}\,.$$

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- the expected number of customers leaving the service station in unit time:

$$\mu\,(1 - p_0) = \mu\,\frac{\rho\,(1-\rho^k)}{1-\rho^{k+1}}$$

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- the expected number of customers in the system $\mathbb{E}[L]$:

$$p_0 \sum_{i=0}^{k} i\rho^i = \rho p_0 \sum_{i=0}^{k} i\rho^{i-1} = \rho p_0 \sum_{i=0}^{k} (\rho^i)' = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}$$

# Overall optimization

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From a social planner's perspective, the goal is to optimize the social welfare (SW), defined here as the total expected net benefit to society.

- the total arrival rate is $\lambda$,
- the probability each customer joins the system is $1 - p_k$,
- each joining customer's net gain is $R - C\mathbb{E}[W]$, where $\mathbb{E}[W]$ is the expected waiting time.

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Note that the payoff for customers that don't join the queue is simply zero. This means that the social welfare (the total payoff per unit time) is

$$\text{SW} = \lambda(1-p_K)\left(R - \frac{1}{1-p_K} C\mathbb{E}[L]\right) = \lambda R(1-p_K) - C\,\mathbb{E}[L].$$

# Overall optimization

$$\mathrm{SW} = \lambda R \, \frac{1-\rho^k}{1-\rho^{k+1}} - C\left(\frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}\right)$$

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Given that SW's dependence on $n$ is "discretely unimodal". In Other words, a local maximum is a global maximum.

- If $n_o$ is the integer that maximizes SW, prove that $n_0$ shall satisfy

$$\frac{n_0(1-\rho) - \rho\,(1-\rho^{n_0})}{(1-\rho)^2} \le \frac{R\,\mu}{C} < \frac{(n_0+1)(1-\rho) - \rho\,(1-\rho^{n_0+1})}{(1-\rho)^2}$$

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Hint: $\text{SW}(n_0) \geq \text{SW}(n_0+1)$ and $\text{SW}(n_0) \geq \text{SW}(n_0-1)$.

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- Prove that $n_0(1-\rho)-\rho\,(1-\rho^{n_0})$ is increasing with $n$.

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- This means we have $n_0 = \lceil v_0 \rceil$, where $v_0$ solves

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Can you see $n_0 \leq n_s$?

Consider the value and derivative of $(v_0\,(1-\rho)-\rho(1-\rho^{v_0}))(1-\rho)^{-2}$ when $v_0=1$.

# Revenue maximization

Imagine that a toll-collecting agency seeks to impose a toll $\theta$, to maximize its revenue rather than optimize the whole system.

How to calculate the total revenue?

# Revenue maximization

Imagine that a toll-collecting agency seeks to impose a toll $\theta$, to maximize its revenue rather than optimize the whole system. How to calculate the total revenue?

- Each joining customer will bring a profit of $\theta$,
- With price $\theta$, each customer will join only when

$$R - \theta - C\frac{i}{\mu} \geq 0,$$

where $i$ is the number of customers in the system.
Thus, there exists an integer $n$ such that

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$$\mathrm{M} = \lambda R \, \frac{1-\rho^{n}}{1-\rho^{n+1}} \left(1 - \frac{n}{v_s}\right),$$

where $v_s = \frac{R\mu}{C}$ as defined before.

We transform the above question to:
Can you show that $n_r$ that maximize M is $n_r = \lfloor v_r \rfloor$, where $v_r$ satisfies

$$v_r + \frac{(1-\rho^{v_r-1})(1-\rho^{v_r+1})}{\rho^{v_r-1}\,(1-\rho)^2} = v_s\,?$$

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Compare $v_r$ with $v_s$.

# Thank You!

ThanQueue