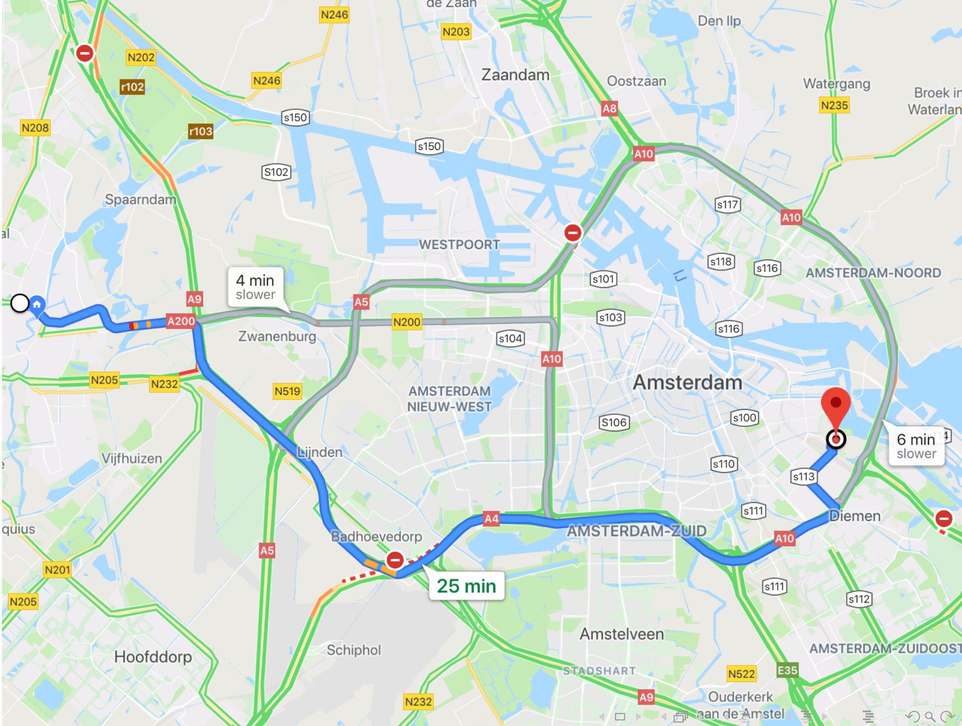
Is it smart to be strategic (on the road)?

Artem Tsikiridis

Centrum Wiskunde & Informatica

NETWORKS goes to school, 2024

We all have the tendency to be "strategic" when making decisions.



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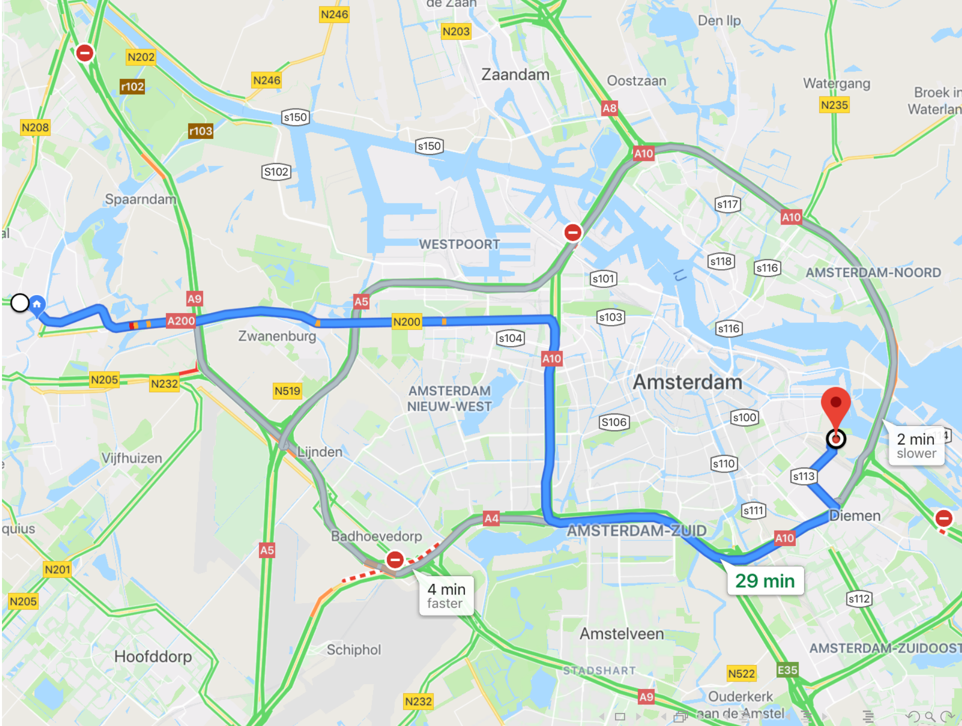

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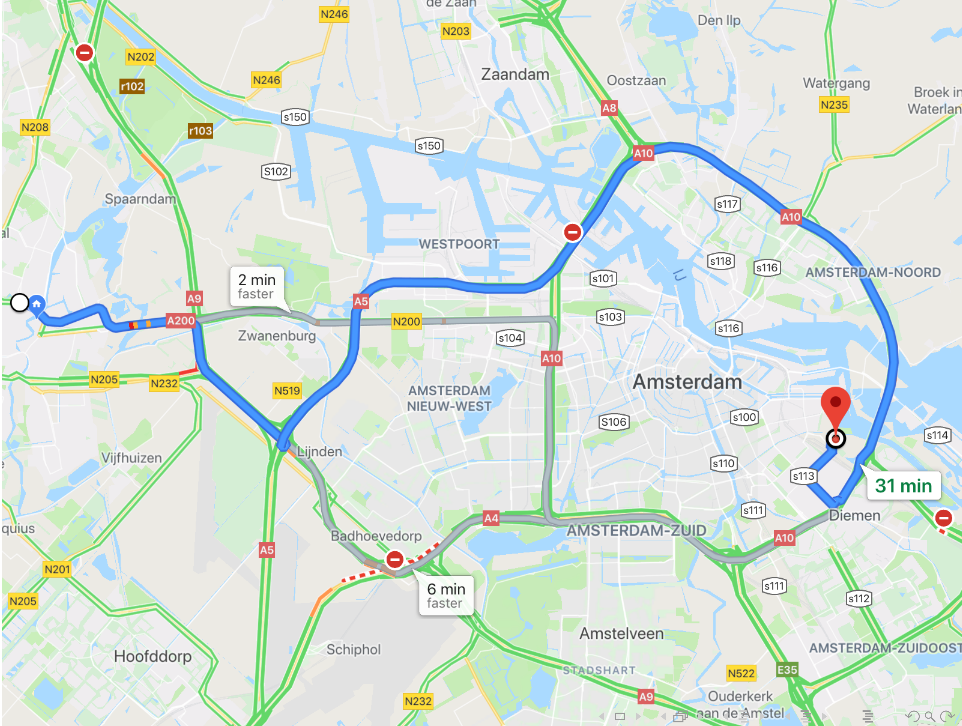

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What is the impact on society if we all make strategic decisions?

<u>This lecture</u>:

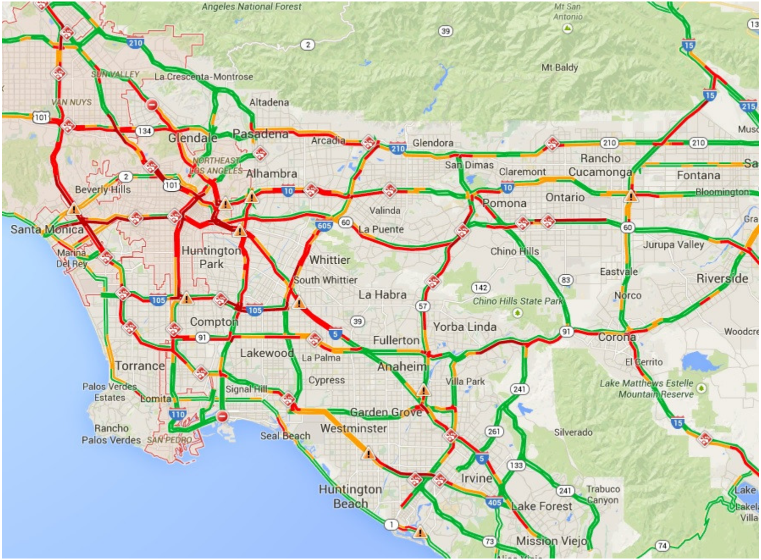
Address this question through the lens of game theory.

What is the impact on society if we all make strategic decisions as drivers?

A selfish perspective?

“Where is everybody going?”

Traffic in Los Angeles



Access to real-time traffic information

Our Goals for Today

- Understand a mathematical model of traffic
- Analyze the model using tools from Game Theory
- Design interventions to improve traffic

Q: Is computer science about computers?

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A Computational Lens on the World

A: Quite a bit, but...

Computer science is no more about computers than astronomy is about telescopes.

E. Dijkstra

A bit controversial, but definitely true for theoretical computer science.

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From Computers to the Internet

- Another "artifact" that computer scientists are studying.
- Computer scientists now need to design algorithms that are robust to strategic manipulation of the input,
- Game Theory has the toolbox to do that!

The Selfish Routing Model we will study is one of the early "big hits" of Algorithmic Game Theory!

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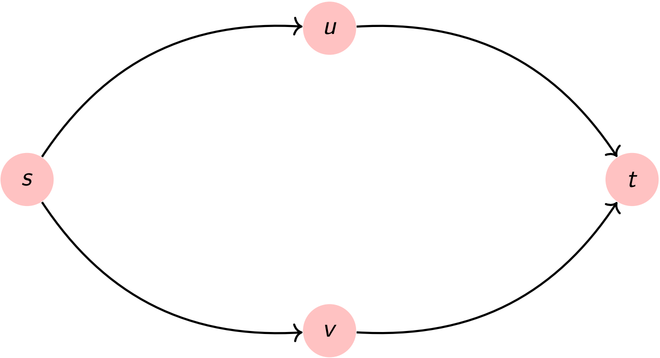
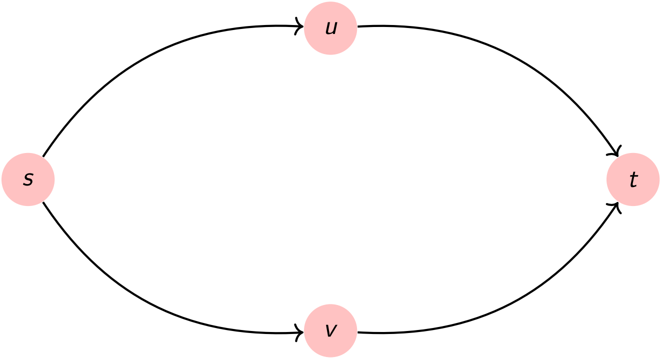
Selfish Routing

A Road Network



- A directed graph $G = (V, E)$ where:
  - V is set of vertices (cities or intersections)
  - E is the set of edges (roads). Can be parallel (multigraph).
- All drivers start trip at $s \in V$ and end at $t \in V$.

A Road Network



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Traffic as a Network Flow

Q: How many cars are there in a city?

A: Well... Many!
But each car causes a proportional amount of traffic.

For large N, if there are N cars, each driver causes a traffic of $1/N$. We model traffic as a continuous flow.

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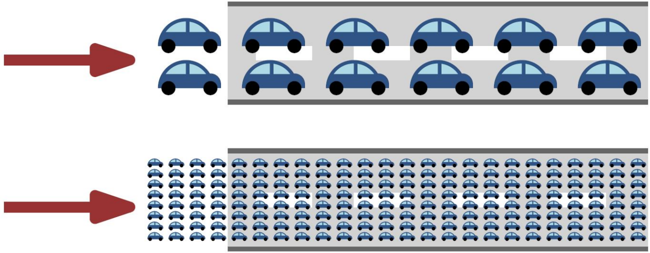

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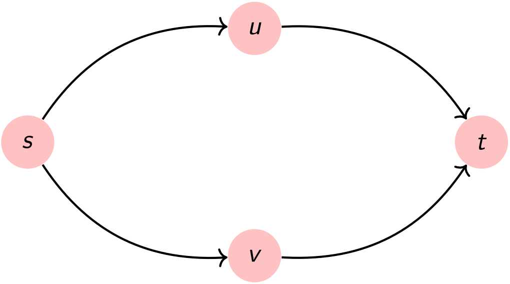
A Selfish Routing Game



- A directed graph $G = (V, E)$ where:
  - V is set of vertices (cities or intersections)
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- We need to transfer a flow of 1 from s to t (all the drivers in the city).

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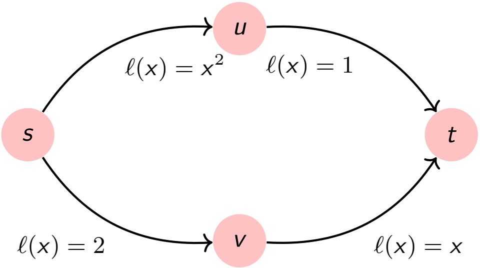

- A directed graph $G = (V, E)$ where:
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- A flow of 1 from s to t (all the drivers in the city) must be sent.
- Every edge $e \in E$ has a latency function ℓ_e representing the travel time. **Example:** If half the cars in the city take (s, u), their travel time will be $\ell(\frac{1}{2}) = \frac{1}{4}$.
- Full information!

Q: Why is this a strategic game?

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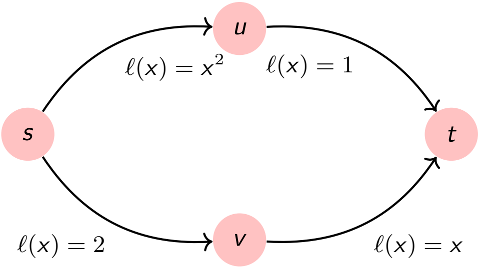

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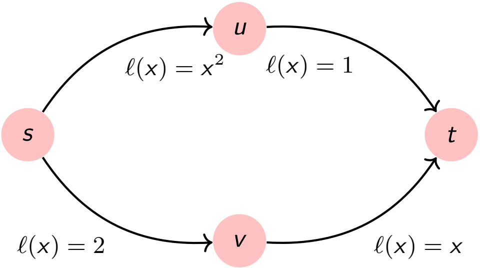

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Feasible Flows and Social Cost

If there are k paths from s to t, let $P_1, \ldots, P_k$ be these paths.

Feasible Flow: A vector of numbers $f = (f_{P_1}, \ldots, f_{P_k})$ such that:

1. For $i = 1, \ldots, k$, it holds that $f_{P_i} \geqslant 0$ (non-negative flow).
2. $\sum_{i=1}^{k} f_{P_i} = 1$ (flow conservation).

Total flow on Edge: Given an edge $e \in E$, the total flow it contains is the sum of flow on all paths that include it! We call it f_e.

Latency of Path: Given a feasible flow f, we define for every $s \to t$ path P, its total latency

$$\ell_P(f) = \sum_{e \in P} \ell_e(f_e)$$

Social Cost: Given a feasible flow f, we define its social cost (or average travel time)

$$SC(f) = \sum_{i=1}^{k} \ell_{P_i}(f) f_{P_i}$$

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Example: Computing the Social Cost of Flows

Keep in Mind!

Path Latency: $\ell_P(f) = \sum_{e \in P} \ell_e(f_e)$

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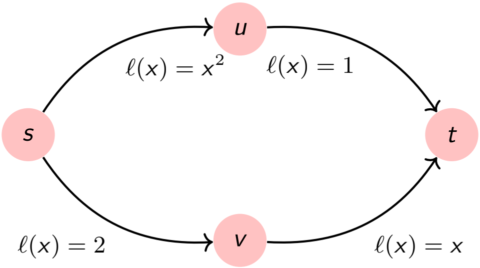

- There are $k = 2$ paths from s to t in this network: $s \to u \to t$ (let's call it P_1) and $s \to v \to t$ (let's call it P_2).
- Sending a flow of $1/2$ on P_1 and a flow of $1/2$ on P_2 is feasible! Why?
- For $f = (1/2, 1/2)$, the total latency on P_1 is

$$\ell_{P_1}\left(\frac{1}{2}, \frac{1}{2}\right) = \ell_{(s,u)}\left(\frac{1}{2}\right) + \ell_{(u,t)}\left(\frac{1}{2}\right) = \frac{1}{4} + 1 = \frac{5}{4}.$$

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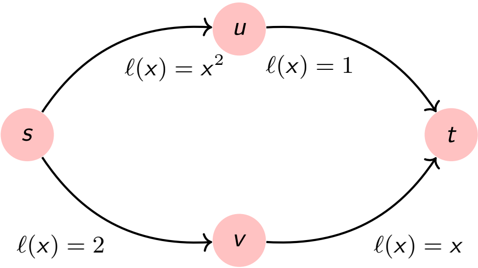

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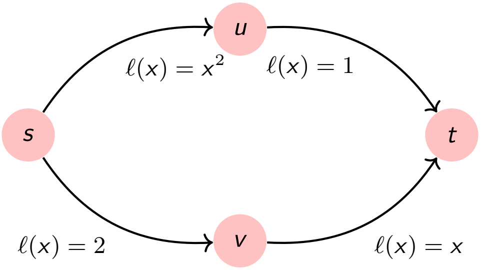

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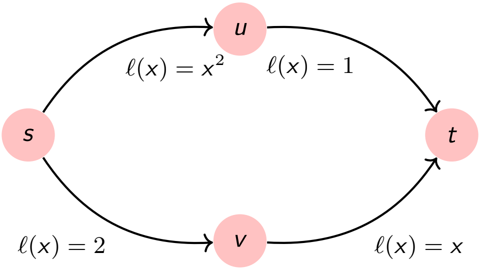

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Example: Computing the Social Cost of Flows (2)

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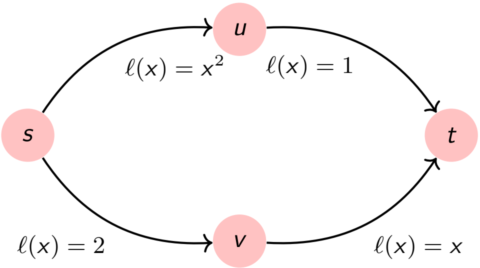

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- The social cost of $f = (1/2, 1/2)$ is

$$SC\left(\frac{1}{2}, \frac{1}{2}\right) = \ell_{P_1}\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \frac{1}{2} + \ell_{P_2}\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \frac{1}{2} = \frac{15}{8} = 1.875$$

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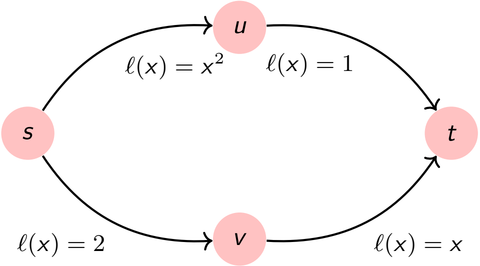

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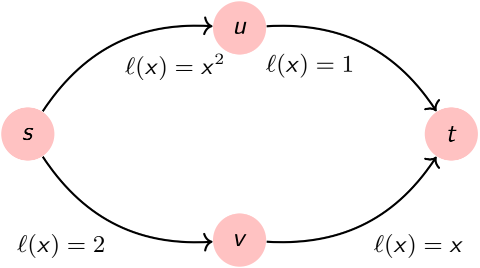


1. Compute the social cost of $f = (1/3, 2/3)$.
2. Compute the social cost of $f = (0, 1)$.
3. Given an $x \in [0, 1]$, write the social cost of $f = (x, 1 - x)$ as a function of x.

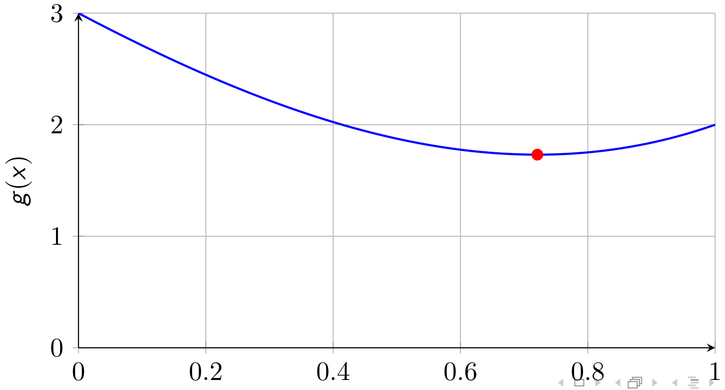
The Optimal Flow

Definition

For a selfish routing game, we say that a feasible flow f^* is optimal if it minimizes the social cost. Formally, for every feasible flow f, it holds that $SC(f^*) \leqslant SC(f)$.

In our example:

$$SC(x, 1-x) = g(x) = x^3 + x^2 - 3x + 3$$



Computing The Optimal Flow

Question

What is the optimal flow $f^* = (x^*, 1 - x^*)$ analytically?

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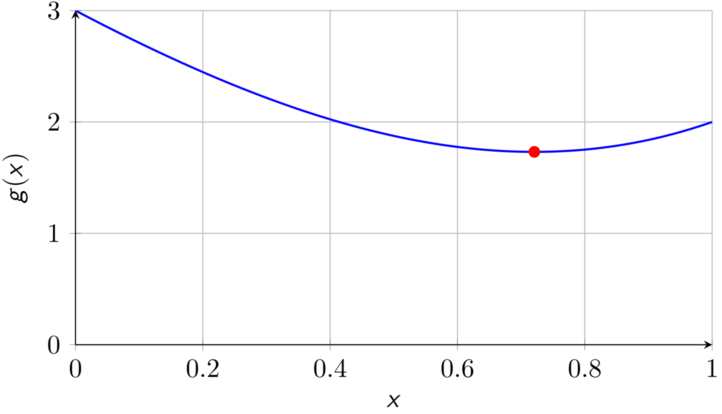

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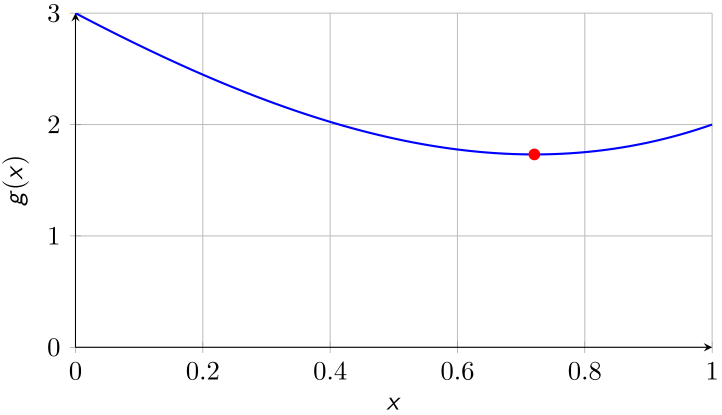

We have that $g'(x) = 3x^2 + 2x - 3$. $x^* \in [0, 1]$ is such that $g'(x^*) = 0$.

Why is the Optimal Flow Useful?

Definition

For a selfish routing game, we say that a feasible flow f^* is optimal if it minimizes the social cost. Formally, for every feasible flow f, it holds that $SC(f^*) \leqslant SC(f)$..

Q: Why is the optimal flow useful?

A: Good benchmark of performance for the social planner of this network.

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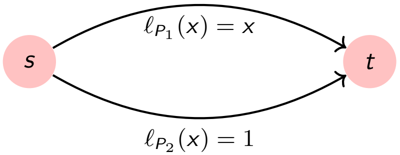
The Nash Flow and the Price of Anarchy

The Pigou Network

The network below was first studied by Arthur Cecil Pigou (1877-1959).



- Two $s \to t$ paths: the upper path (P_1) and the lower path (P_2).
- **Path P_1:** Travel time increases linearly with the number of cars. Generally fast, unless crowded.
- **Path P_2:** Travel not depending on cars using it! High congestion even for a few cars (old/narrow road)!

Q: What is the optimal flow f^* for the Pigou Network? What is the optimal social cost $SC(f^*)$?

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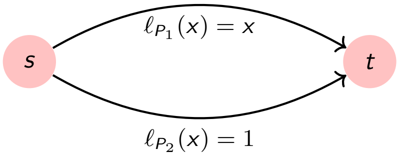

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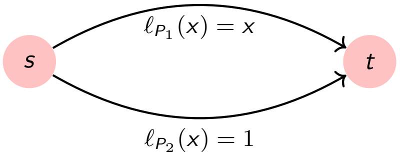

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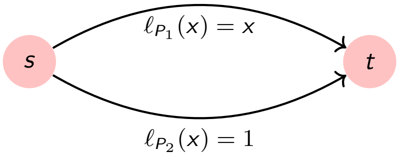

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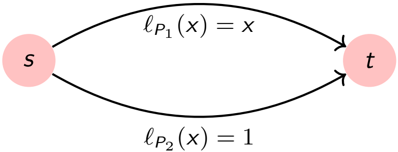

A: Given an $x \in [0,1]$, the social cost of the flow $(x, 1-x)$ is

$$SC(x, 1-x) = \ell_{P_1}(x, 1-x) \cdot x + \ell_{P_2}(x, 1-x) \cdot (1-x)$$
$$= x \cdot x + 1 \cdot (1-x) = x^2 + 1 - x.$$

The function $g(x) = x^2 + 1 - x$ is minimized for $x^* = 1/2$ and $SC(1/2, 1/2) = 3/4$. Optimal flow f^* is

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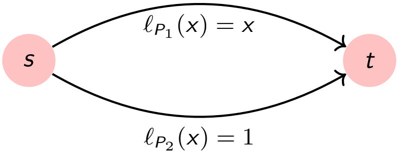

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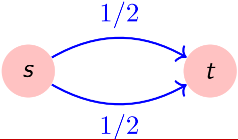
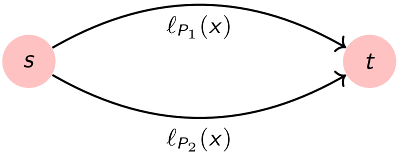


The Perspective of a driver on P_2



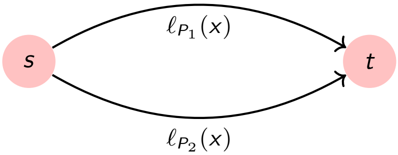
Q: Consider a flow $(x, 1-x)$ with $x > 0$. What is the perspective of a selfish driver on P_2 who wants to minimize her travel time?

A: They are regretting being on P_2 unless $\ell_{P_2}(1-x) \leqslant \ell_{P_1}(x)$ (equilibrium condition).

Not possible!

When $\ell_{P_1}(x) = x$ and $\ell_{P_2}(x) = 1$, the above inequality is true only for $x = 1$. Therefore, $(1, 0)$ is an equilibrium flow!

The Perspective of a driver on P_2



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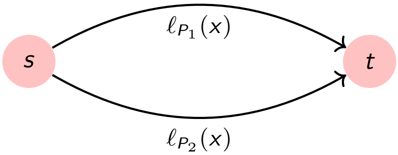

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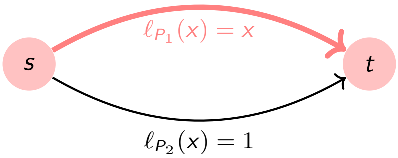

Definition

For a selfish routing game, we say that a feasible flow f_{nash} is a Nash flow (or equilibrium flow) if, for every pair of $s \to t$ paths P_i and P_j with $f_{P_i} > 0$ and $f_{P_j} > 0$, it holds that

$$\ell_{P_i}(f_{\text{nash}}) = \ell_{P_j}(f_{\text{nash}}).$$

For our example, the Nash flow is $f_{\text{nash}} = (1, 0)$ and $SC(1, 0) = \ell_{P_1}(1, 0) \cdot 1 = 1$.

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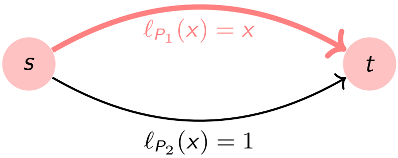

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Price of Anarchy

Measuring Selfishness: The optimal Social Cost for the Pigou network is $\frac{3}{4}$. On the other hand, the Social Cost of the Nash flow is 1! Selfishness increases social by 33%.

Also,

$$\frac{SC(f_{\text{nash}})}{SC(f^*)} = \frac{4}{3}$$

Definition

The Price of Anarchy for a class of selfish routing games is the worst-case ratio of the social cost at equilibrium to the optimal social cost.

Observation:

We have computed a lower bound on the Price of Anarchy for selfish routing games. Surprisingly, this is also the worst-case for a very broad class of games: games with latency functions of the form $\ell(x) = \alpha x + \beta$ (affine functions).

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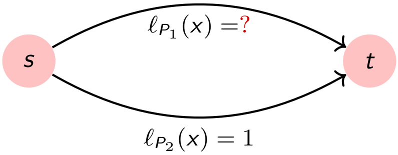
Exercises on Pigou Networks



Compute an optimal flow and a Nash flow when:

1. $\ell_{P_1}(x) = \frac{x^2+x}{2}$.
2. $\ell_{P_1}(x) = x^3$.
3. $\ell_{P_1}(x) = x^d$, for a given positive integer d.

The Social Planner Strikes Back!

Tolls?

A tax on drivers or a tool for the social planner to align incentives?

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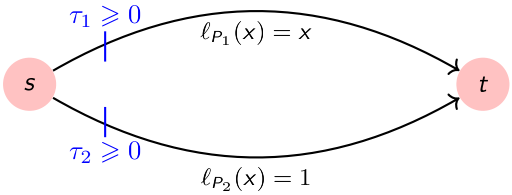
Tolls on the Pigou Network



Main idea: Give an incentive to drivers to drive on P_2!

Assumption 1: Social planner is not interested in profit, but in decreasing the average travel time (social cost) of the equilibrium flow!

Definition

For a selfish routing game with a monetary toll τ_e for each $e \in E$, the adjusted latency (or cost) for a flow f and each $e \in E$ is:

$$c_e(f_e) = \ell_e(f_e) + \gamma \cdot \tau_e,$$

where $\gamma > 0$ is a parameter of how "sensitive" drivers are to tolls.

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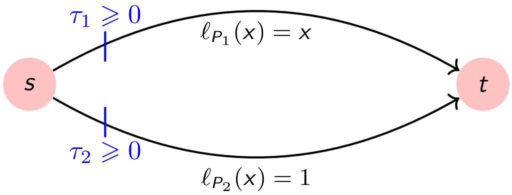

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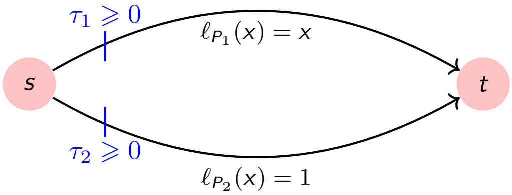

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where $\gamma > 0$ is a parameter of how "sensitive" drivers are to tolls.

Goal: Set tolls $\tau_1, \tau_2 \geqslant 0$ so that $f^* = (\frac{1}{2}, \frac{1}{2})$ becomes a Nash flow.

By the definition of the Nash flow, it must be that

$$c_{P_1}\left(\frac{1}{2}, \frac{1}{2}\right) = c_{P_2}\left(\frac{1}{2}, \frac{1}{2}\right).$$

Equivalently,

$$\ell_{P_1}\left(\frac{1}{2}\right) + \gamma \cdot \tau_1 = \ell_{P_2}\left(\frac{1}{2}\right) + \gamma \cdot \tau_2 \iff \frac{1}{2} + \gamma\tau_1 = 1 + \gamma\tau_2 \iff \tau_1 = \frac{1}{2\gamma} + \tau_2.$$

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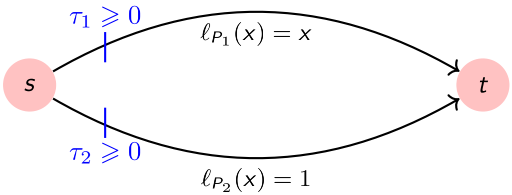
Exercise on Tolls

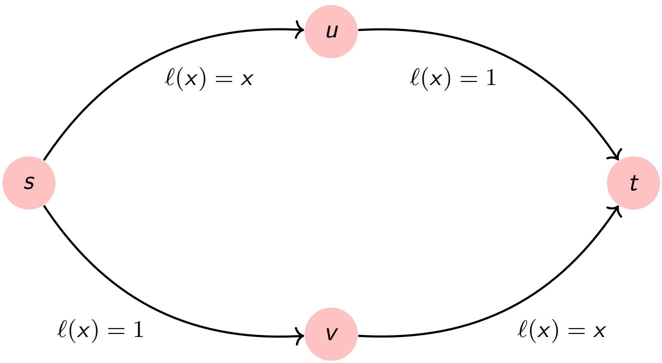


Find a pair of tolls $\tau_1^*, \tau_2^* \geqslant 0$ so that:

1. The flow $f^* = (\frac{1}{2}, \frac{1}{2})$ is a Nash flow.
2. The revenue of the government under f^* is minimized.

Adding a new road

Another Network



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Q: Do you see why?

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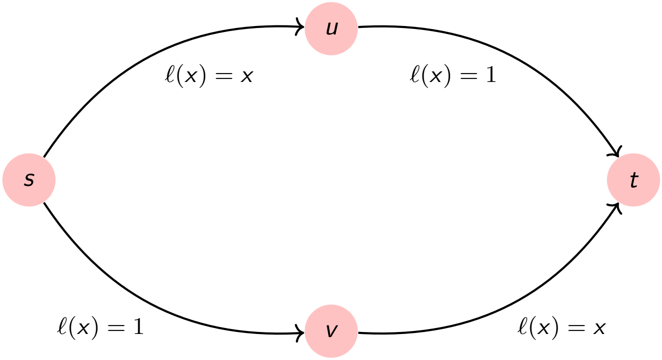

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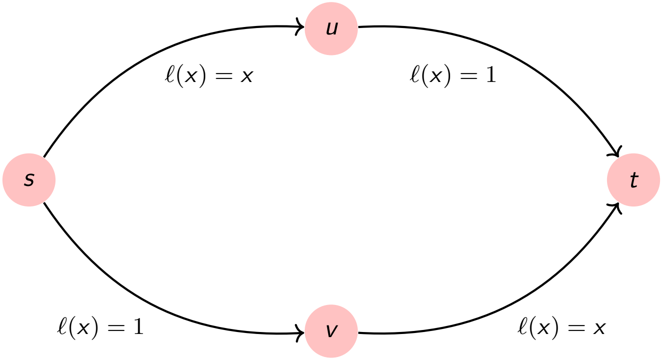

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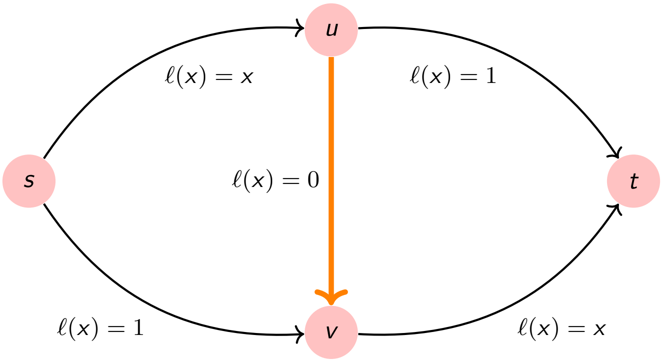
Adding a New Highway



The government decides to build a very fast road from $u \to v$ to improve the social cost of $\frac{3}{2}$.

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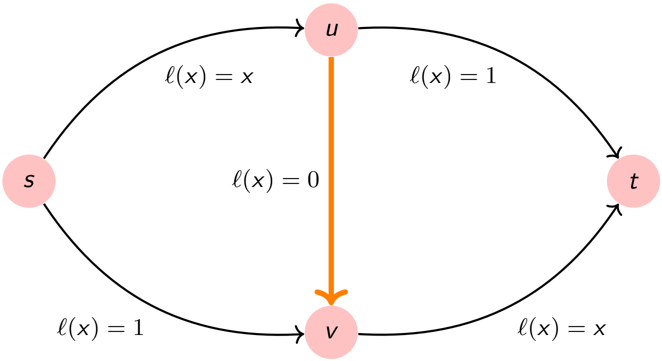

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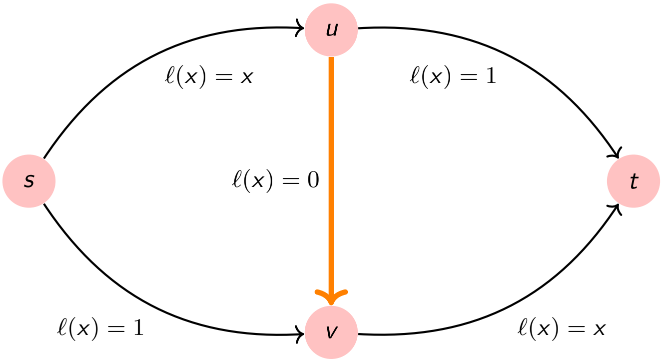

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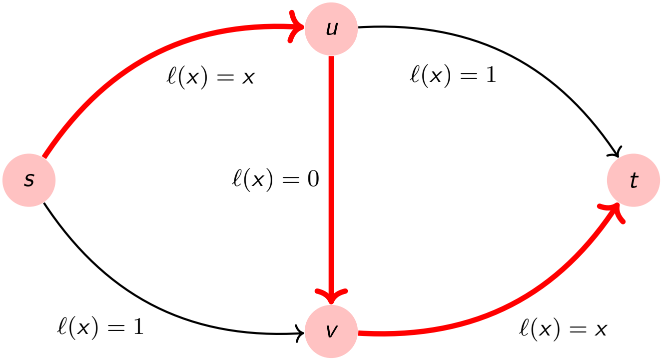
The Braess Paradox



A: The path $s \to u \to v \to t$ is used by all cars in the Nash flow!

The social cost of this flow is $2 > \frac{3}{2}$. Average travel time got worse! This is the Braess Paradox (observed first by Prof. Dirichlet Braess in 1968).

See also: `https://youtu.be/RmLrpci_tfo` (simulation) and `https://youtu.be/Cg73j3QYRJc` (spring paradox)

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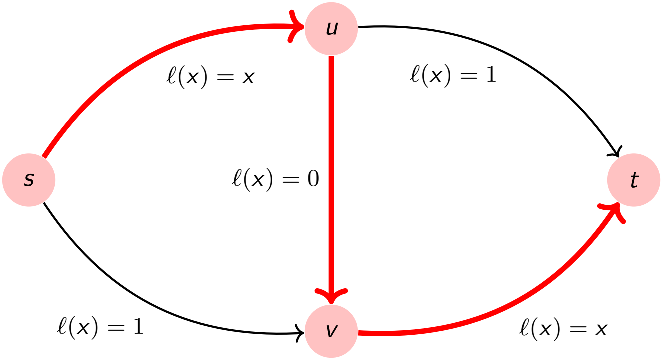

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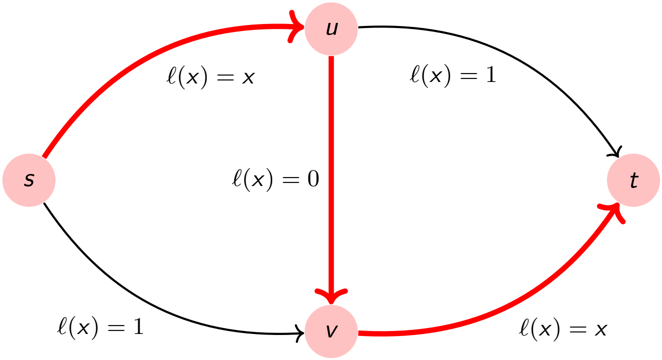

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The New York Times
nytimes.com

December 25, 1990

What if They Closed 42d Street and Nobody Noticed?

By GINA KOLATA

ON Earth Day this year, New York City's Transportation Commissioner decided to close 42d Street, which as every New Yorker knows is always congested. "Many predicted it would be doomsday," said the Commissioner, Lucius J. Riccio. "You didn't need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem."

But to everyone's surprise, Earth Day generated no historic traffic jam. Traffic flow actually improved when 42d Street was closed.

To mathematicians, this may be a real-world example of Braess's paradox, a statistical theorem that holds that when a network of streets is already jammed with vehicles, adding a new street can make traffic flow even more slowly. Seeking Out a New Street

The reason is that in crowded conditions, drivers will pile into a new street, clogging both it and the streets that provide access to it. By the same token, removing a major thoroughfare may actually ease congestion on the streets that normally provide access to it. And because other major streets are already overcrowded, diverting still more traffic to them may not make much difference.

The New York Times
nytimes.com

December 25, 1990

What if They Closed 42d Street and Nobody Noticed?

By GINA KOLATA

ON Earth Day this year, New York City's Transportation Commissioner decided to close 42d Street, which as every New Yorker knows is always congested. "Many predicted it would be doomsday," said the Commissioner, Lucius J. Riccio. "You didn't need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem."

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Dr. Joel E. Cohen, a mathematician at Rockefeller University in New York, says the paradox does not always hold; each traffic network must be analyzed on its own. When a network is not congested, adding a new street will indeed make things better. But in the case of congested networks, adding a new street probably makes things worse at least half the time, mathematicians say.

Dr. Cohen and Dr. Frank P. Kelly of the University of Cambridge in England published the most recent analysis of the traffic paradox in the current issue of The Journal of Applied Probability. In their paper, they show that the paradox occurs when the traffic is described by a sophisticated statistical model. Previous work had used what Dr. Cohen describes as an overly simple and less realistic model.

The traffic paradox was first described in 1968 by Dr. Dietrich Braess of the Institute for Numerical and Applied Mathematics in Munster, Germany. He found that when one street was added to a simple four-street network, all the vehicles took longer to get through.

Dr. Braess's result was "very surprising," said Dr. Richard Steinberg of A.T.&T.'s Bell Laboratories in Holmdel, N.J. Dr. Steinberg and colleagues studied how often the paradox would hold true, and determined in 1983 that "it is just as likely to occur as not."

He and his colleagues also turned up a paradox of their own: that in some situations, "when you add more delays along a route, more people use it." Honk, Honk

Dr. Cohen and Dr. Kelly have now examined traffic networks with a sophisticated analytic method known as queuing theory, which describes traffic jams in terms of vehicles lining up on the streets. They found a simple traffic network in which adding a street increased travel time.

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He cited a German paper, published in 1969, reporting that the City of Stuttgart had tried to ease downtown traffic by adding a new street. But congestion only got worse, and so, in desperation, the authorities closed the street. Traffic flow improved.

New York's Transportation Commissioner, Mr. Riccio, has a doctoral degree himself (in engineering, from Lehigh University), and he said he favored using mathematical models to try to improve traffic flow. "I believe in these models," he said, and added that he would welcome a call from Dr. Cohen to discuss how his work could apply to New York City's daunting traffic problems.

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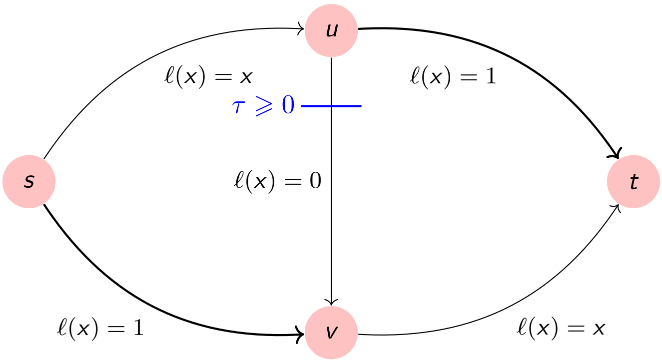
Exercise: Fixing the Braess Paradox



1. Compute the social cost of sending a flow of $1/3$ on each path.
2. Given a $\gamma > 0$, what should $\tau \geqslant 0$ be so that the above flow is a Nash flow?

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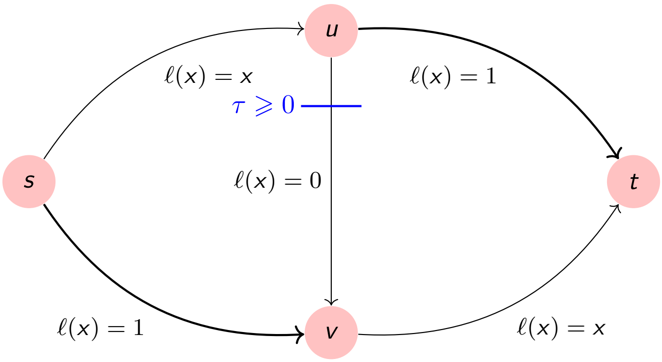

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Conclusions

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A: Oftentimes not!
→ results in inefficient outcomes

But: strategic behavior seems unavoidable

Need: design coordination mechanisms (tolls, carpooling) which incentivize players to reach good outcomes

→ algorithmic game theory is a challenging research field which lies in the intersection of mathematics, computer science and economics

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THANKS FOR YOUR ATTENTION!